Welcome to
e - Learning
on

STATIKA DAN MEKANIKA STRUKTUR 2

PRODI TEKNIK SIPIL FT UKI

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DOSEN PRODI TEKNIK SIPIL UKI

COURSE GOALS

This course has two specific goals:

- (i) To introduce students to concepts of stresses and strains; shearing force and bending moment; torsion and deflection of different structural / machine elements: as well as analysis of stresses in two dimensions

COURSE GOALS

- (ii) To develop theoretical and analytical skills relevant to the areas mentioned in (i) above.

COURSE OUTLINE

Unit -I Stress, strain and deformation of solids

Simple and compound bars, thermal stresses, Elastic constants and strain energy course outline …..

Unit – II Beams – loads and stresses

shear force and bending moments, theory of simple bending, bending and shear stresses

course outline ….

Unit –III Torsion

Torsion on circular bars, power transmitted by solid and hollow shafts – springs course outline ….

Unit – IV Deflection of beams and column theories

Double integration method, Macaulay’s method and area moment method.

Columns – Euler’s theory and Rankine’s formula

Unit – V Stresses in two dimensions

Thin cylindrical and spherical shells, Principal stresses and planes – Mohr's circle

UNIT -1

STRESS, STRAIN AND DEFORMATION OF SOLIDS

Definitions:

Rigid Body: A **rigid body** is an idealization of a solid body of finite size in which deformation is neglected.

In other words, the distance between any two given points of a rigid body remains constant in time regardless of external forces exerted on it.

Deformable body: A **deformable body** is a physical body that deforms, meaning it changes its shape or volume while being acted upon by an external force.

Elasticity : The property which enables a material to change its dimension, volume, or shape in direct response to a force effecting such a change and to recover its original form upon the removal of the force.

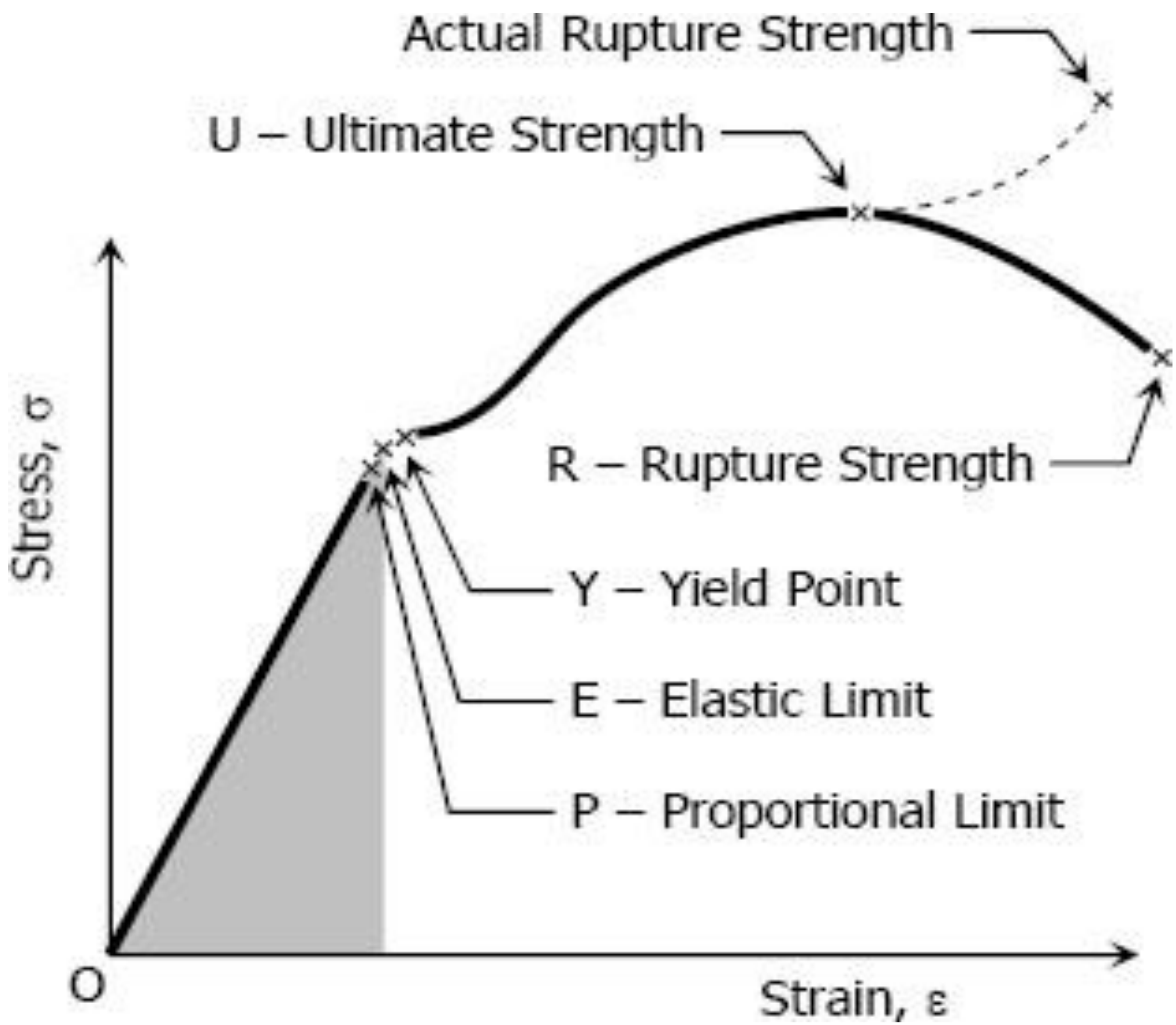
Elastic : If a material returns to its original size and shape on removal of load causing deformation, it is said to be **elastic**.

Elastic limit, is the limit up to which the material is perfectly elastic.

Elastic stress is the maximum stress or force per unit area within a solid material that can arise before the onset of permanent deformation.

- When stresses up to the elastic limit are removed, the material resumes its original size and shape.

- Stresses beyond the elastic limit cause a material to yield or flow. For such materials the elastic limit marks the end of elastic behaviour and the beginning of plastic behaviour.



Actual Rupture Strength
U – Ultimate Strength
R – Rupture Strength
Y – Yield Point
E – Elastic Limit
P – Proportional Limit
Stress, σ
O
Strain, ε

The strength of any material relies on the following three terms:

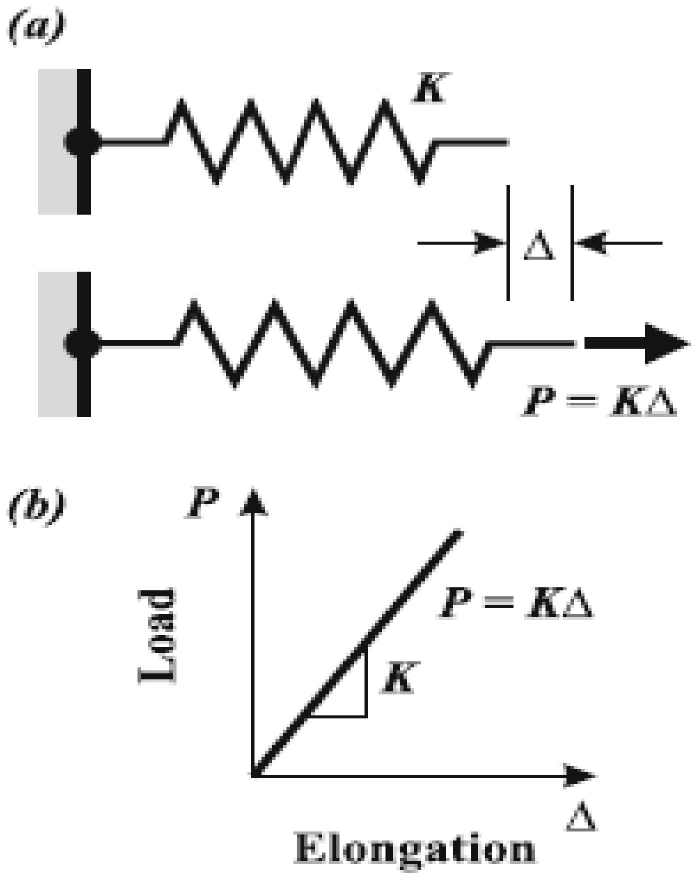
Strength,**Stiffness** and **Stability**.

Strength means load carrying capacity, or it is the ability to withstand an applied stress without failure. The applied stress may be tensile, compressive, or shear.

Stiffness means resistance to deformation or elongation, and

Stability means ability to maintain its initial configuration.

Stiffness (k)



Stress is a measure of the internal reaction between elementary particles of a material in resisting separation, compaction, or sliding that tend to be induced by external forces.

Mathematically, it is expressed as the ratio of the load applied to the cross sectional area.

Unit: Usually N/m^2 (Pa), N/mm^2, MN/m^2, GN/m

- **Note:** $1\ N/mm^2 = 1\ MN/m^2 = 1\ MPa$

Strain is defined as the amount of deformation an object experiences compared to its original size and shape.

- i.e., strain is the relative change in shape or size of an object due to externally applied force.
- Mathematically, it is expressed as the ratio of change in dimension to original dimension
- i.e strain = Δ/L

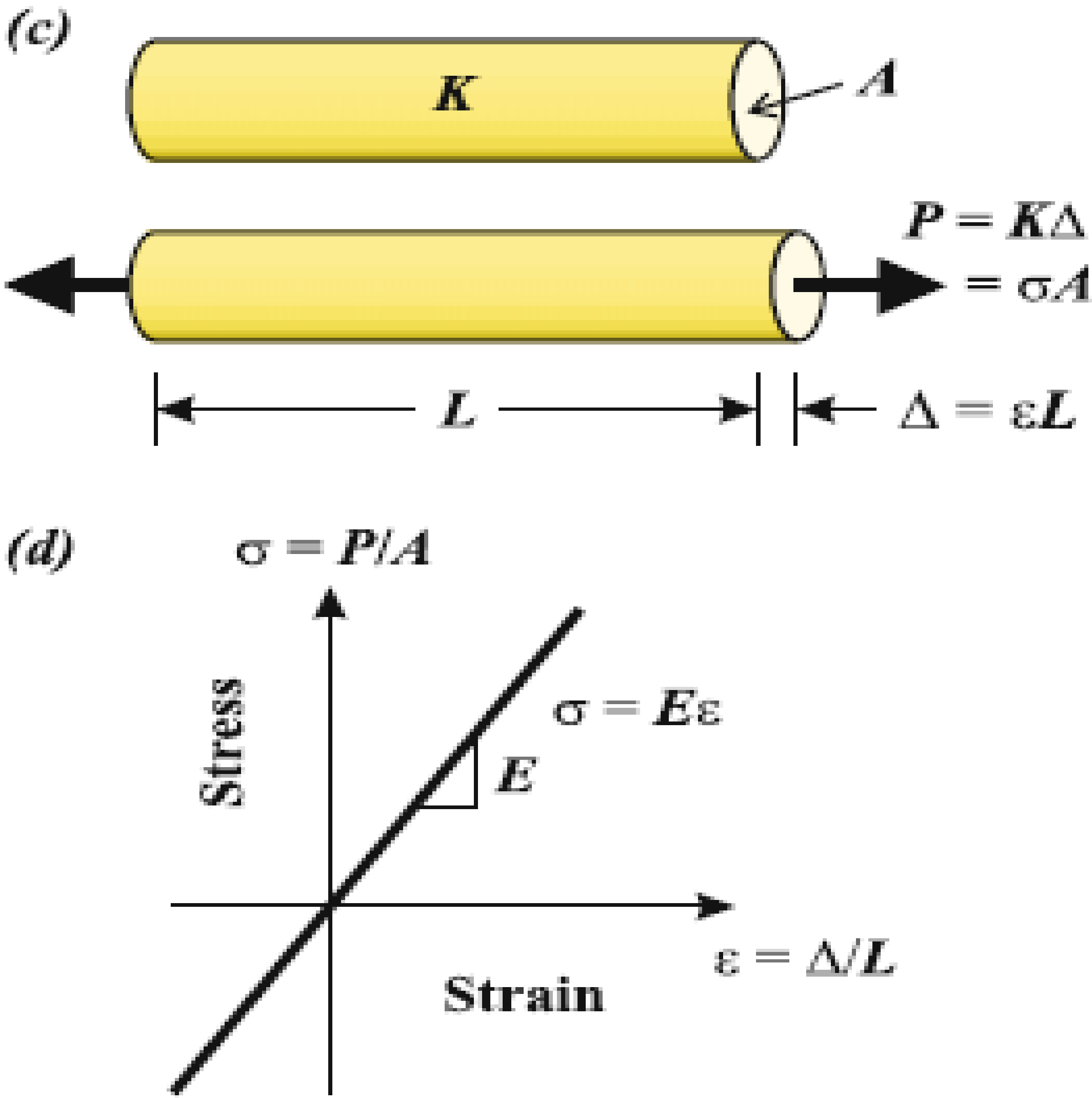
Hooke’s Law: It states that providing the limit of proportionality of a material is not exceeded, the stress is directly proportional to the strain produced.

- i.e., with in the elastic limit, the stress is directly proportional to the strain
- If a graph of stress and strain is plotted as load is gradually applied, the first portion of the graph will be a straight line.

- **Modulus of Elasticity (E)**
- It is defined as the stress intensity required by the body to produce unit strain.
- It is a measure of the stiffness of a material.

$$E = \frac{\text{Direct stress } (\sigma)}{\text{Direct strain } (\varepsilon)}$$

Units: N/m^2 (Pa), or N/mm^2, or MN/m^2, or GN/m^2 etc.,


(c)
K
A
P = KΔ
= σA
L
Δ = εL
(d)
σ = P/A
Stress
σ = Eε
E
ε = Δ/L
Strain

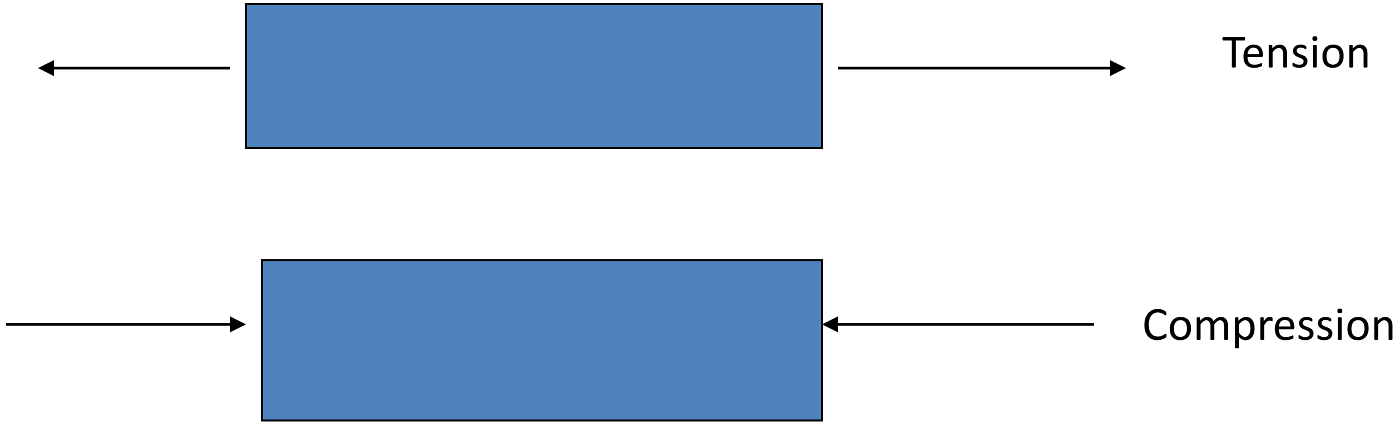
Direct or Normal stress

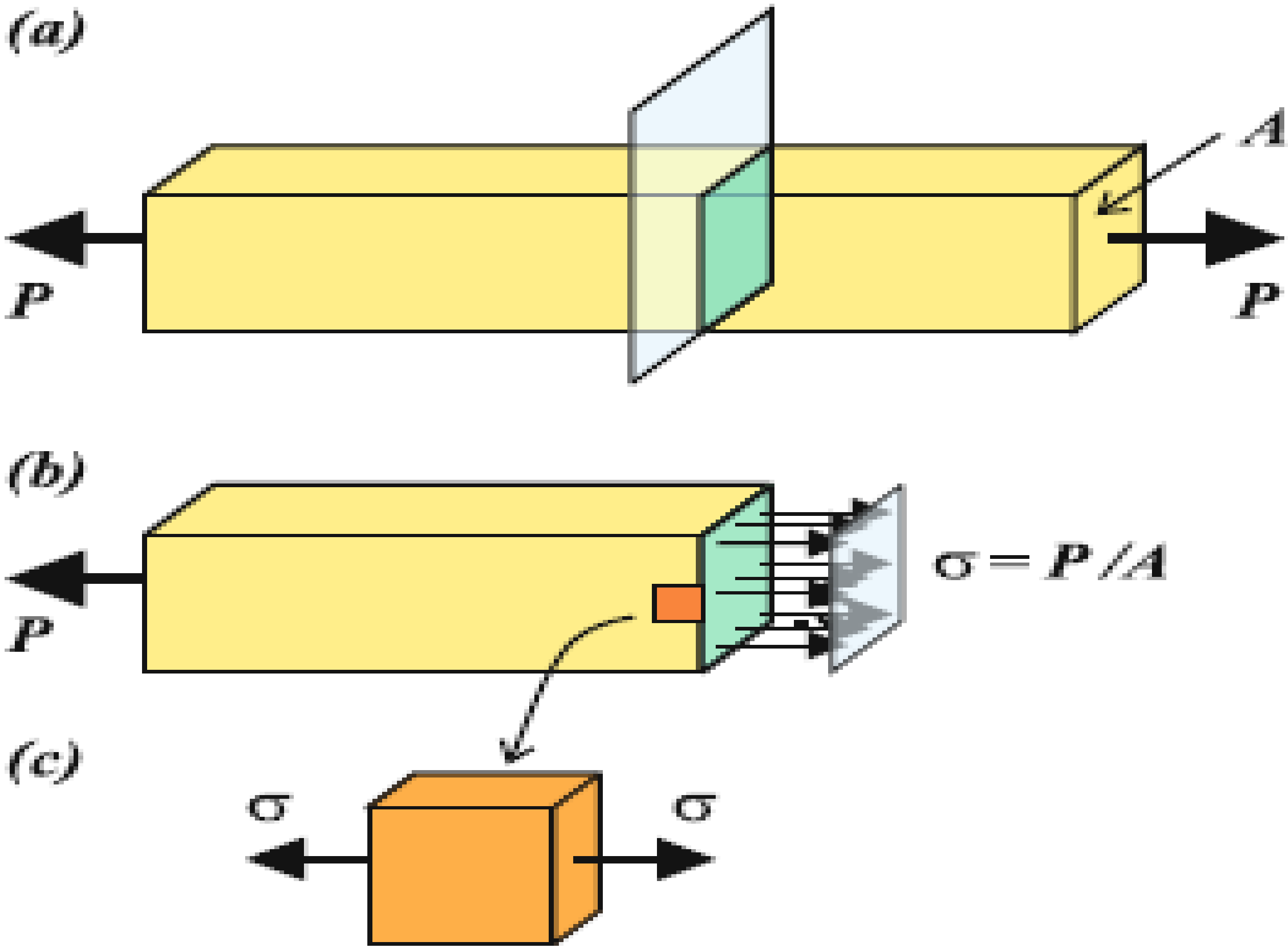
When a force is transmitted through a body, the body tends to change its shape or deform. The body is said to be strained.

- Direct Stress = $\dfrac{\text{Applied Force (F)}}{\text{Cross Sectional Area (A)}}$

Direct Stress Cont...

- Direct stress may be tensile, σ_t or compressive, σ_c and result from forces acting perpendicular to the plane of the cross-section

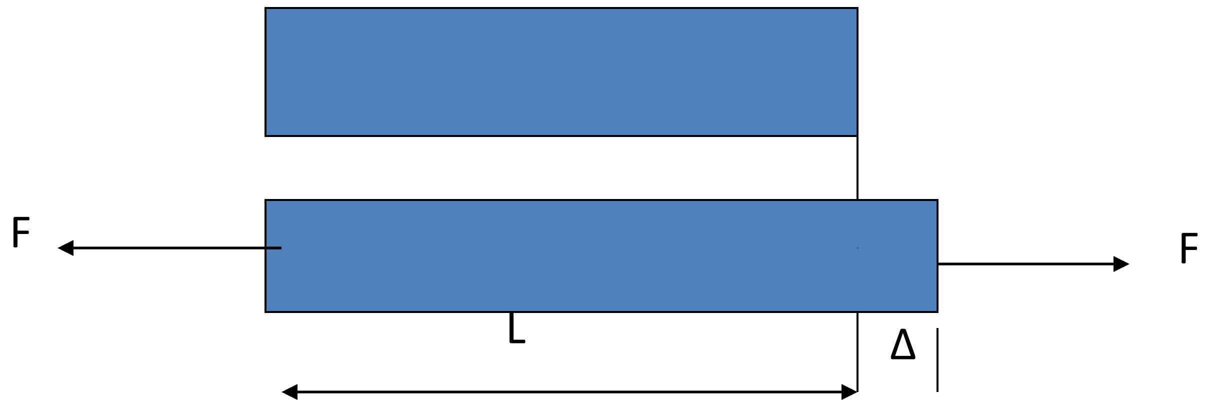



(a)
A
P
P
(b)
σ = P / A
P
(c)
σ
σ

Direct or Normal Strain

- When loads are applied to a body, some deformation will occur resulting to a change in dimension.
- Consider a bar, subjected to axial tensile loading force, F. If the bar extension is Δ and its original length (before loading) is L, then tensile strain is:

Direct or Normal Strain Cont….



- Direct Strain (ε) $= \dfrac{\text{Change in Length}}{\text{Original Length}}$

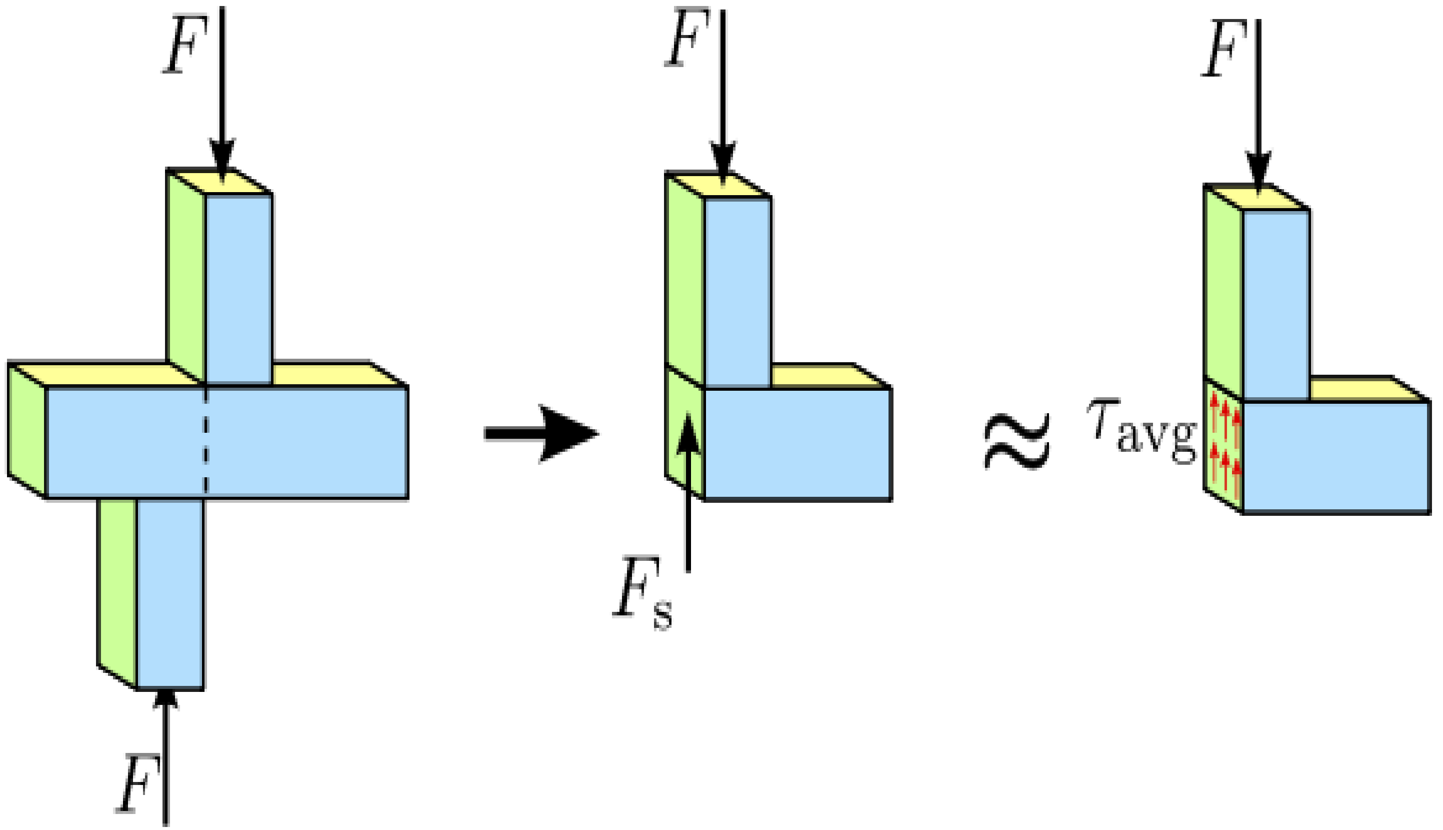
i.e. $\varepsilon = \Delta/L$

Direct or Normal Strain Contd.

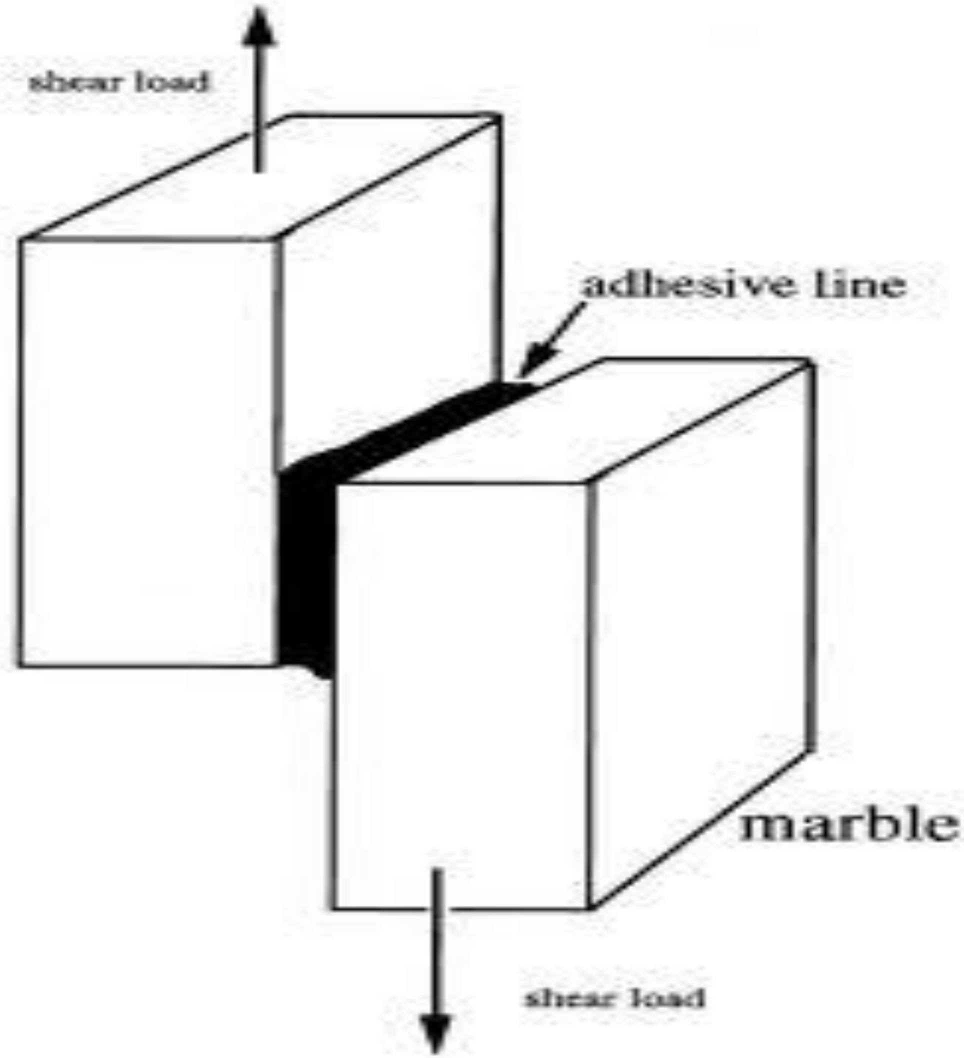
- As strain is a ratio of lengths, it is dimensionless.
- Similarly, for compression by amount, Δ : Compressive strain = - Δ /L
- **Note:** Strain is positive for an increase in dimension and negative for a reduction in dimension.

Shear Stress and Shear Strain

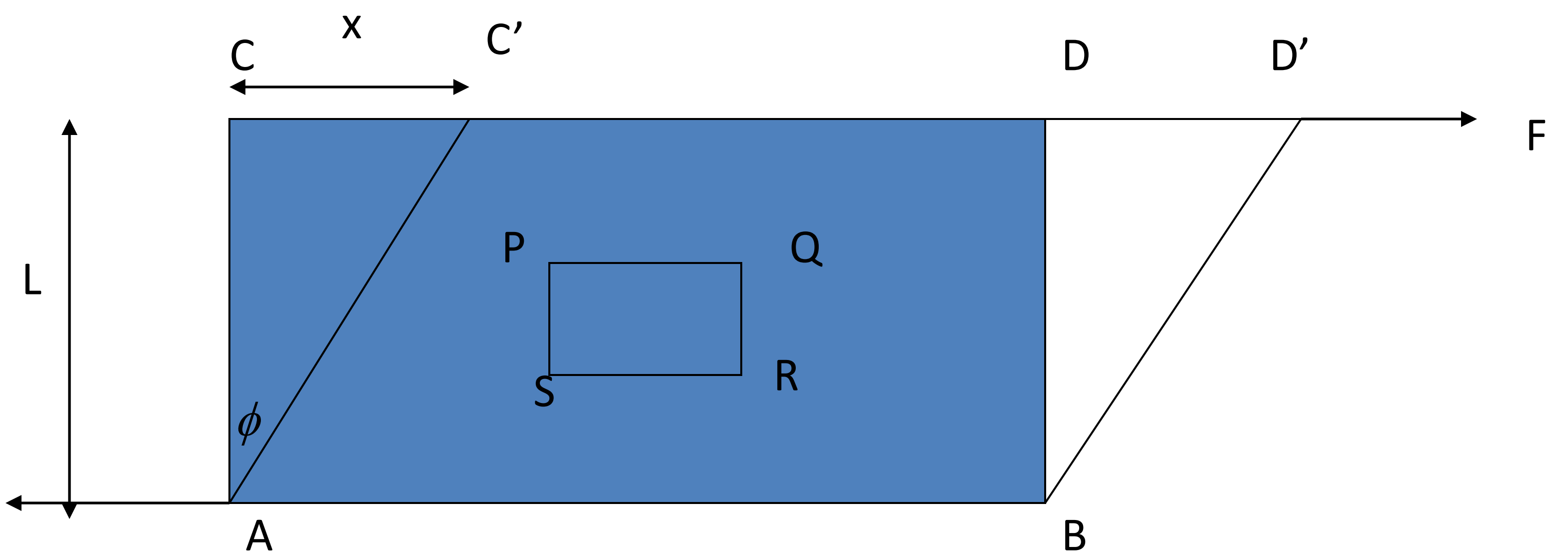
- Shear stresses are produced by equal and opposite parallel forces not in line.
- The forces tend to make one part of the material slide over the other part.
- Shear stress is tangential to the area over which it acts.



F
F
F_s
F
F
$\approx$
τ_{avg}


shear load
adhesive line
marble
shear load

Shear Stress and Shear Strain Contd.



Shear Stress and Shear Strain Contd.

Shear strain is the distortion produced by shear stress on an element or rectangular block as above. The shear strain, (gamma) is given as:

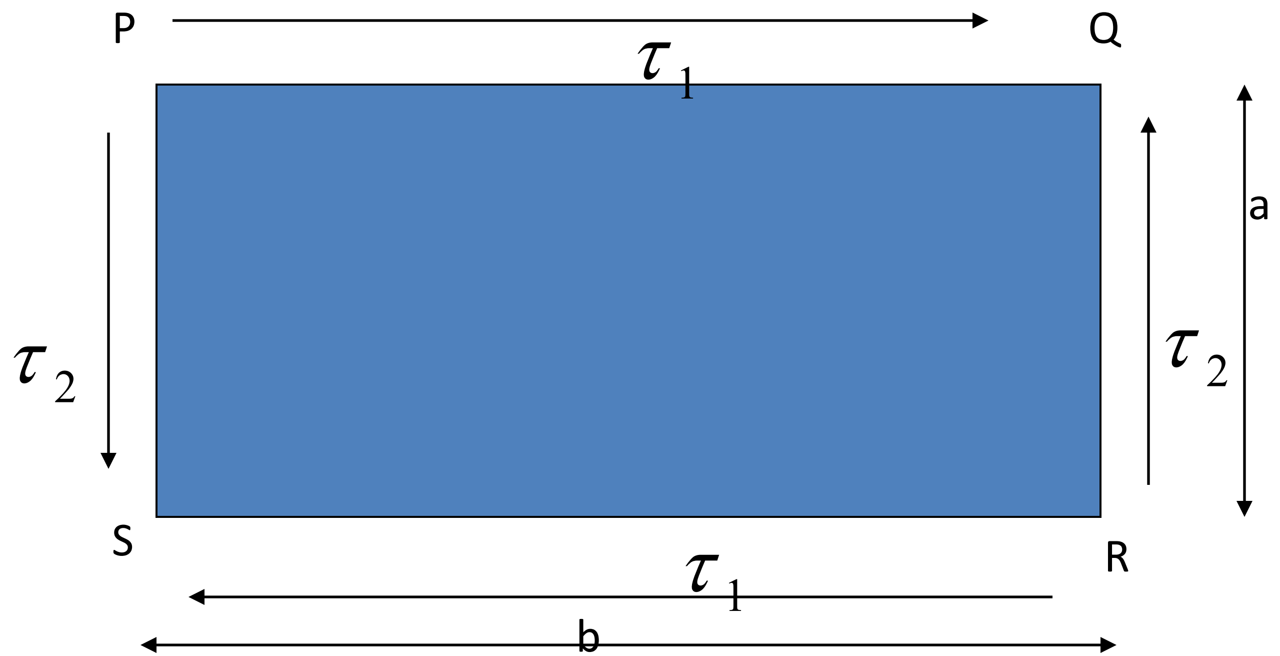
$$\gamma = x/L$$

$$\gamma = \tan\phi$$

Shear Stress and Shear Strain Contd

- For small angle ϕ, $\gamma = \phi$
- Shear strain then becomes the change in the right angle.
- It is dimensionless and is measured in radians.

Complementary Shear Stress



Consider a small element, PQRS of the material in the last diagram. Let the shear stress created on faces PQ and RS be τ_1

Complimentary Shear Stress Contd.

- The element is therefore subjected to a couple and for equilibrium, a balancing couple must be brought into action.
- This will only arise from the shear stress on faces QR and PS.
- Let the shear stresses on these faces be

$$\tau_2$$

Complimentary Shear Stress Contd.

- Let t be the thickness of the material at right angles to the paper and lengths of sides of element be a and b as shown.
- For equilibrium, clockwise couple = anticlockwise couple
- i.e. Force on PQ (or RS) x a = Force on QR (or PS) x b
- $$\tau_1 \quad x \quad b\,t \quad x \quad a \quad = \tau_2 \quad x \quad a\,t \quad x \quad b$$

$$i.e. \quad \tau_1 \quad = \quad \tau_2$$

- **Thus:** Whenever a shear stress occurs on a plane within a material, it is automatically accompanied by an equal shear stress on the perpendicular plane.

  The direction of the complementary shear stress is such that their couple opposes that of the couple due to the original shear stresses.

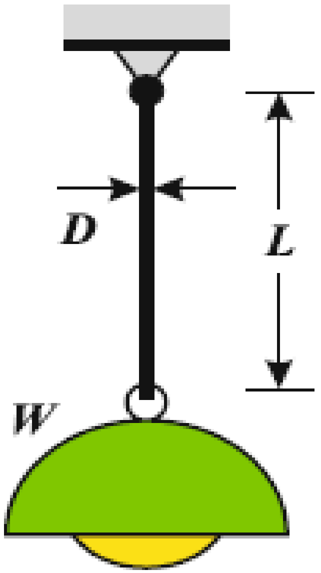
Deformation of Simple Bars

We know, modulus of elasticity,

$$E = \frac{\sigma}{\epsilon} = \frac{P/_A}{\Delta/_l} = \frac{Pl}{A\Delta}$$

$$\therefore Deformation, \Delta = \frac{Pl}{AE}$$

Ex 1: A lamp weighing $W = 50N$ *hangs from* the ceiling by a steel wire of diameter $D = 2.5mm$ Determine the elongation Δ of the wire due to the lamp's weight.



Given:

The wire supporting the lamp *is 1.5m long*

The wire is steel with elastic modulus *E = 200 GPa*

Required: Determine the elongation Δ of the wire due to the lamp's weight.

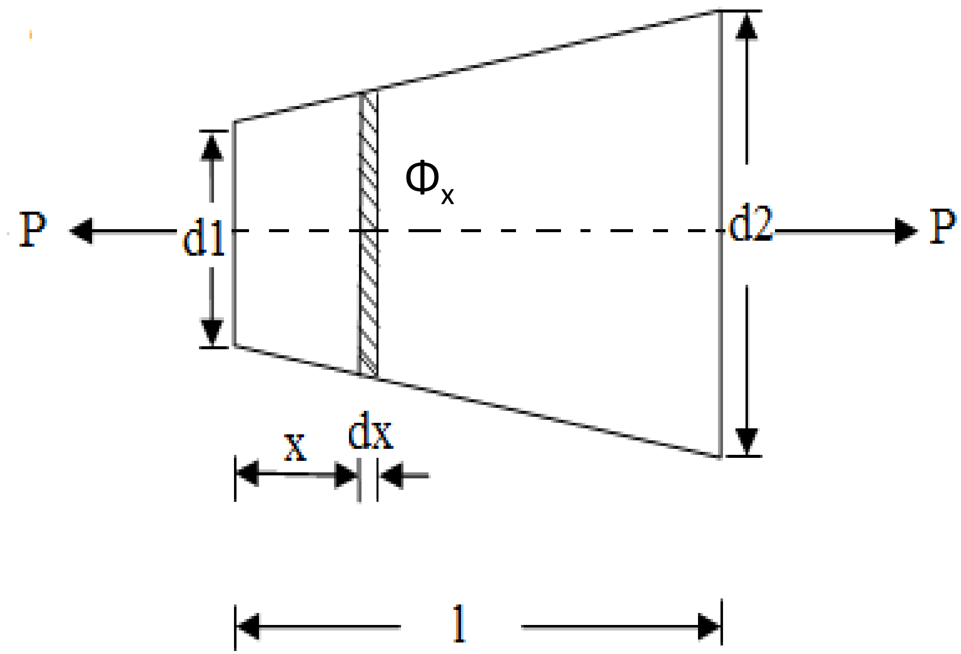
Solution :

Area = 4.91 mm^2

Δ = (50 x 1500) / (4.91 x 200 000)

= 0.0764 mm

Deformation of a solid truncated conical Bar



Deformation of the element,

$$\delta\Delta = \frac{P\,dx}{A_x E} = \frac{P\,dx}{\frac{\pi}{4}(\emptyset_x{}^2)E} = \frac{4P\,dx}{\pi\,(d_1 + kx)^2 E}$$

Where

$$\emptyset_x = d_1 + \frac{(d_2 - d_1)}{l}\,x = d_1 + kx$$

where, $$k = \frac{d_2 - d_1}{l}$$

Total deformation of the bar,

$$\Delta = \int_0^l \delta\Delta$$

$$= \int_0^l \frac{4P\,dx}{\pi E(d_1 + kx)^2}$$

$$= \frac{4P}{\pi E}\int_0^l (d_1 + kx)^{-2}\,dx$$

$$= \frac{4P}{\pi E}\left(\frac{-1}{k}\right)\left[\frac{1}{(d_1 + kx)}\right]_0^l$$

$$= - \frac{4P}{\pi E K}\left[\frac{1}{\left(d_1+\frac{(d_2-d_1)}{l}\right)x}\right]_0^l$$

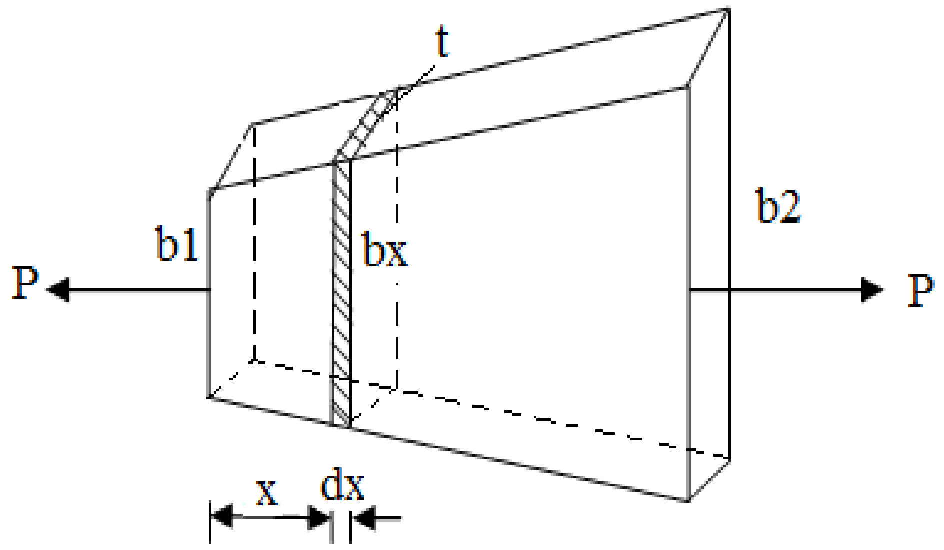
$$= - \frac{4P}{\pi E K}\left[\frac{1}{d_2} - \frac{1}{d_1}\right]$$

$$= \frac{4P(d_2-d_1)}{\pi E \frac{(d_2-d_1)d_1 d_2}{l}}$$

$$= \frac{4Pl}{\pi E d_1 d_2}$$

$$\Delta = \frac{4Pl}{\pi E d_1 d_2}$$

Deformation of uniformly tapering rectangular bar



Consider a small element of width dx at a distance x from the left end as shown in Fig.

Deformation of the element,

$$\delta\Delta = \frac{P\,dx}{b_x t\,E}$$

$$= \frac{P\,dx}{(b_1 + kx)t\,E}$$

where, $$k = \frac{b_2 - b_1}{l}$$

Total deformation of the bar,

$$= \frac{P}{tE}\int_{0}^{l}\frac{dx}{(b_1 + kx)}$$

$$= \frac{\mathrm{P}}{\mathrm{ktE}} [\log_{\mathrm{e}} (b_1 + kx)]$$

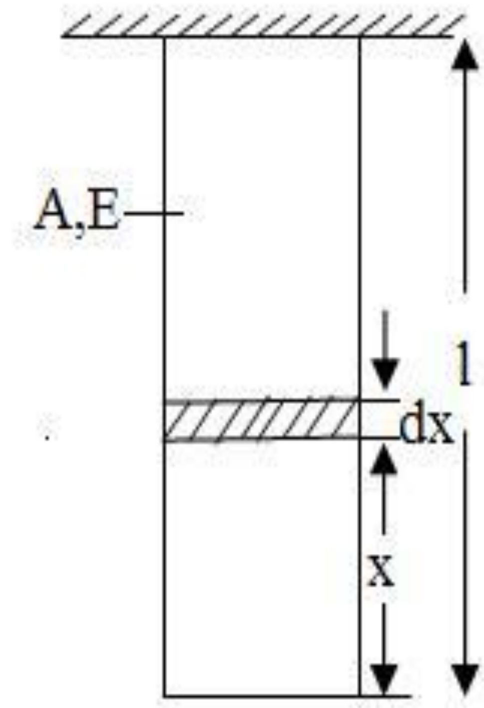
$$= \frac{\mathrm{P}}{\mathrm{ktE}} [\log_{\mathrm{e}} \left(b_1 + \frac{(b_2 - b_1)}{l} x \right)]$$

$$= \frac{\mathrm{P}}{\mathrm{ktE}} [\log (b_2) - \log (b_1)]$$

$$\therefore \Delta = \frac{P}{ktE}[\log(\frac{b_2}{b_1})]$$

$$= \frac{P\,L}{(b_2 - b_1)tE}[\log(\frac{b_2}{b_1})]$$

<u>Elongation of a bar due to its self weight</u>



Deformation of the element,

$$\delta\Delta = \frac{Ax\gamma dx}{AE}$$

$$= \frac{\gamma x dx}{E}$$

Total deformation, $$\Delta = \int_0^l \delta\Delta$$

$$= \int_0^l \frac{\gamma}{E} x dx$$

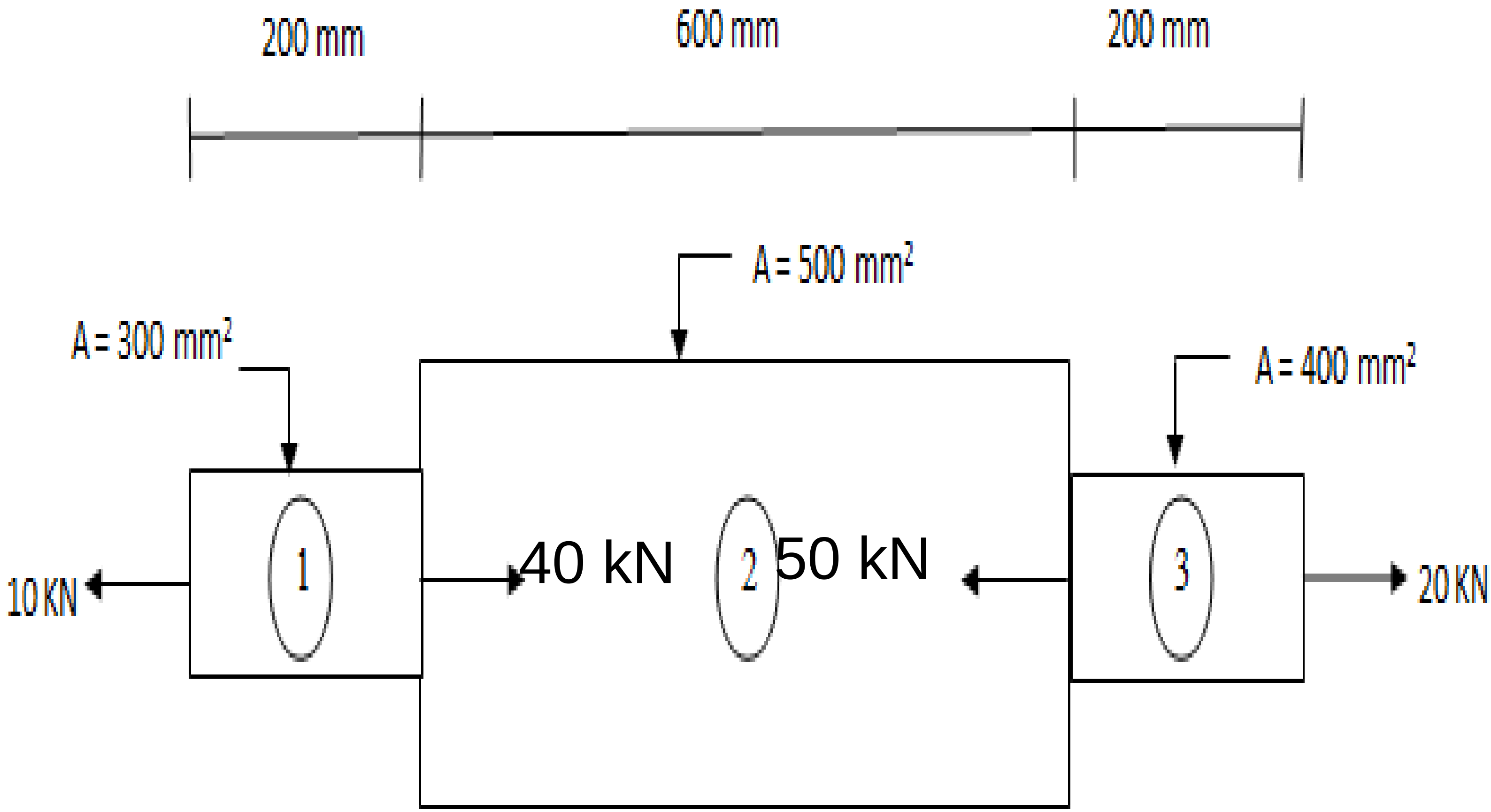
$$= \frac{\gamma l^2}{2E} \times \frac{A}{A}$$

$$\Delta = \frac{Wl}{2AE}$$ where w = total weight of the bar

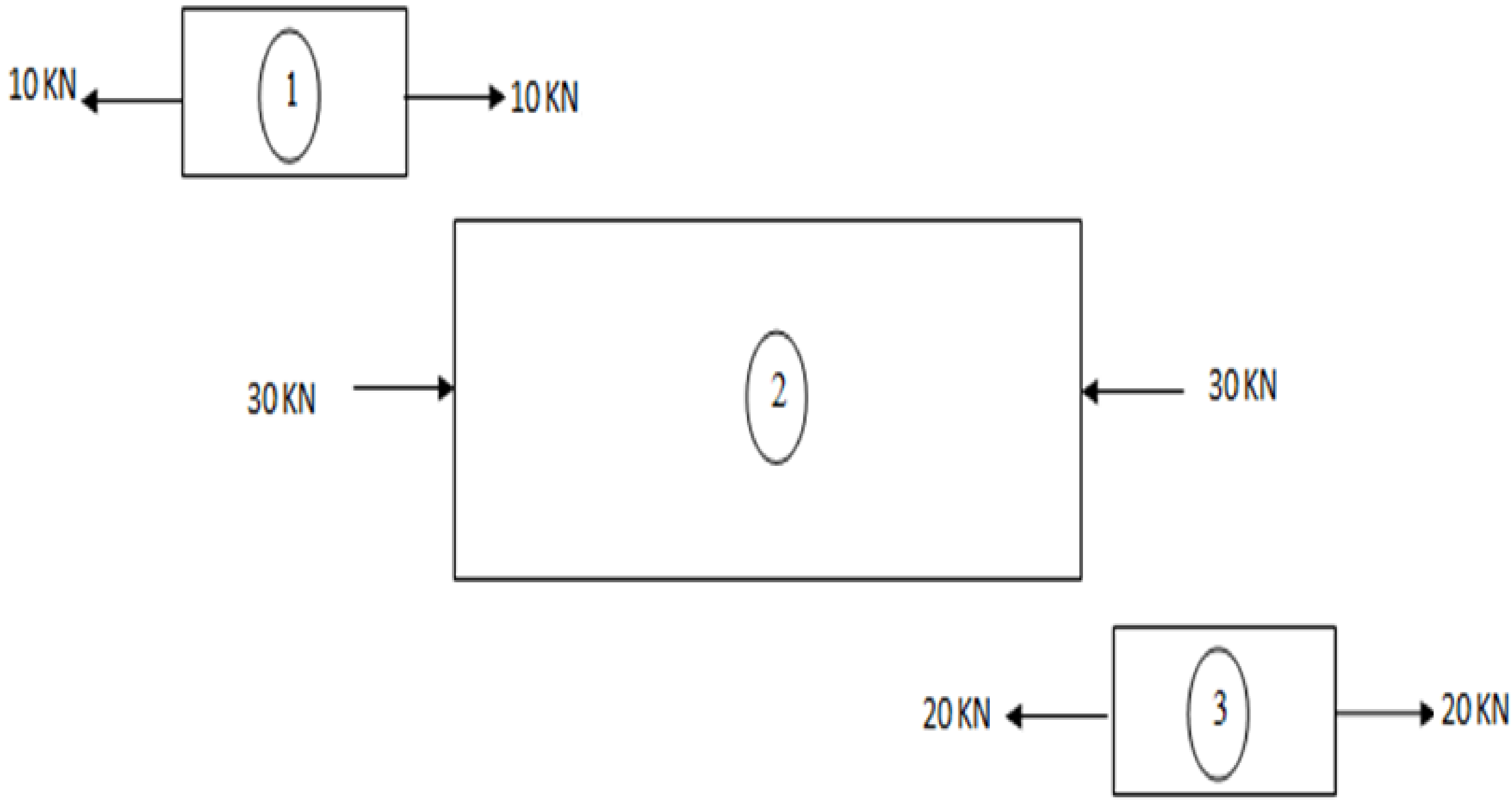
$= Al\gamma$

Deformation of a Stepped Bar

Ex 2: A steel bar ABCD is shown in fig. Determine the total change in length of the bar. Take E = 200 Gpa.



Free body diagram



$$\Delta l = \frac{P_1 l_1}{A_1 E_1} + \frac{P_2 l_2}{A_2 E_2} + \frac{P_3 l_3}{A_3 E_3}$$

$$= \frac{10 \text{ x } 10^3 \text{x } 200}{300 \text{ x } 2 \text{ x } 10^5}$$

$$+ \frac{-30 \text{ x } 10^3 \text{x } 600}{500 \text{ x } 2 \text{ x } 10^5}$$

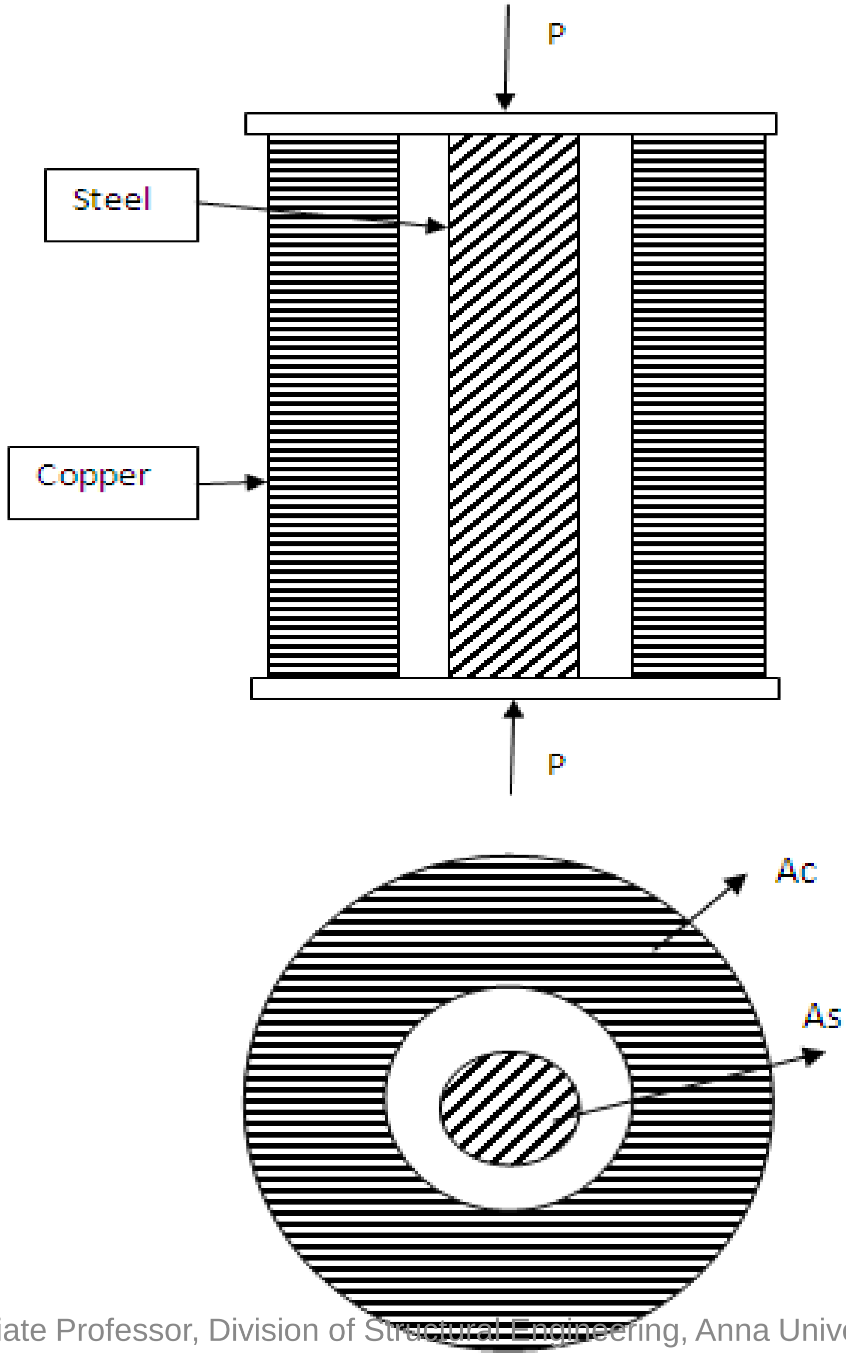
$$+ \quad \frac{20 \text{ x } 10^3 \text{x } 200}{400 \text{ x } 2 \text{ x } 10^5}$$

= 0.0333 - 0.18 + 0.05

= -0.0967 mm (- sign indicates the deformation in contraction)

Composite bar

Definition: A bar made up of two or more different materials and fastened together to prevent uneven straining and act as a single bar subjected to axial loading or temperature change.


P
Steel
Copper
P
Ac
As

by equilibrium condition

$$P = P_S + P_C$$

$$= f_S A_S + f_C A_C \quad \text{------} > (1)$$

by compatibility condition

$$\Delta s = \Delta c$$

$$\frac{P_s l_s}{A_s E_s} = \frac{P_c l_c}{A_c E_c} \Rightarrow \frac{f_s}{E_s} = \frac{f_c}{E_c}$$

$$f_s = \frac{E_s}{Ec} f_c \_\_\_\_\_\_ (2)$$

solving (1) & (2), we get f_s and f_c

Ex 3:

A steel rod of diameter 25 mm is placed inside a copper tube of internal diameter 30 mm and external diameter 40 mm, the ends being firmly fastened together. Determine the stresses induced in the steel rod and copper tube when the composite bar is subjected to a compressive load of 500 kN. Take $E_s = 2 \times 10^5$ N/mm ^{2} and $E_c = 1.1 \times 10^5$ N/mm ^{2}

Solution

Area of steel rod $A_S = \frac{\pi}{4} \times 25^2 = 490.63mm^2$

Area of copper tube $A_c = (\pi/4) \times (40^2 - 30^2)$

$$= 549.78mm^2$$

By equilibrium equation,

$$P = f_s A_s + f_c A_c$$

$$490.63 f_s + 549.78 f_c = 500 \times 10^3 \quad \ldots\ldots\ldots(1)$$

By compatibility condition, $e_s = e_c$

$$f_s = \frac{E_S}{E_c} \times f_c \quad ie., f_s = \frac{2.1 \times 10^5}{1.1 \times 10^5} \times f_c$$

$$= 1.91 f_c \_\_\_\_\_\_ (2)$$

Solving (1) and (2) , we get

$$f_s = 642.28 \mathrm{N}/mm^2 \quad f_c = 336.27 \mathrm{N}/mm^2$$

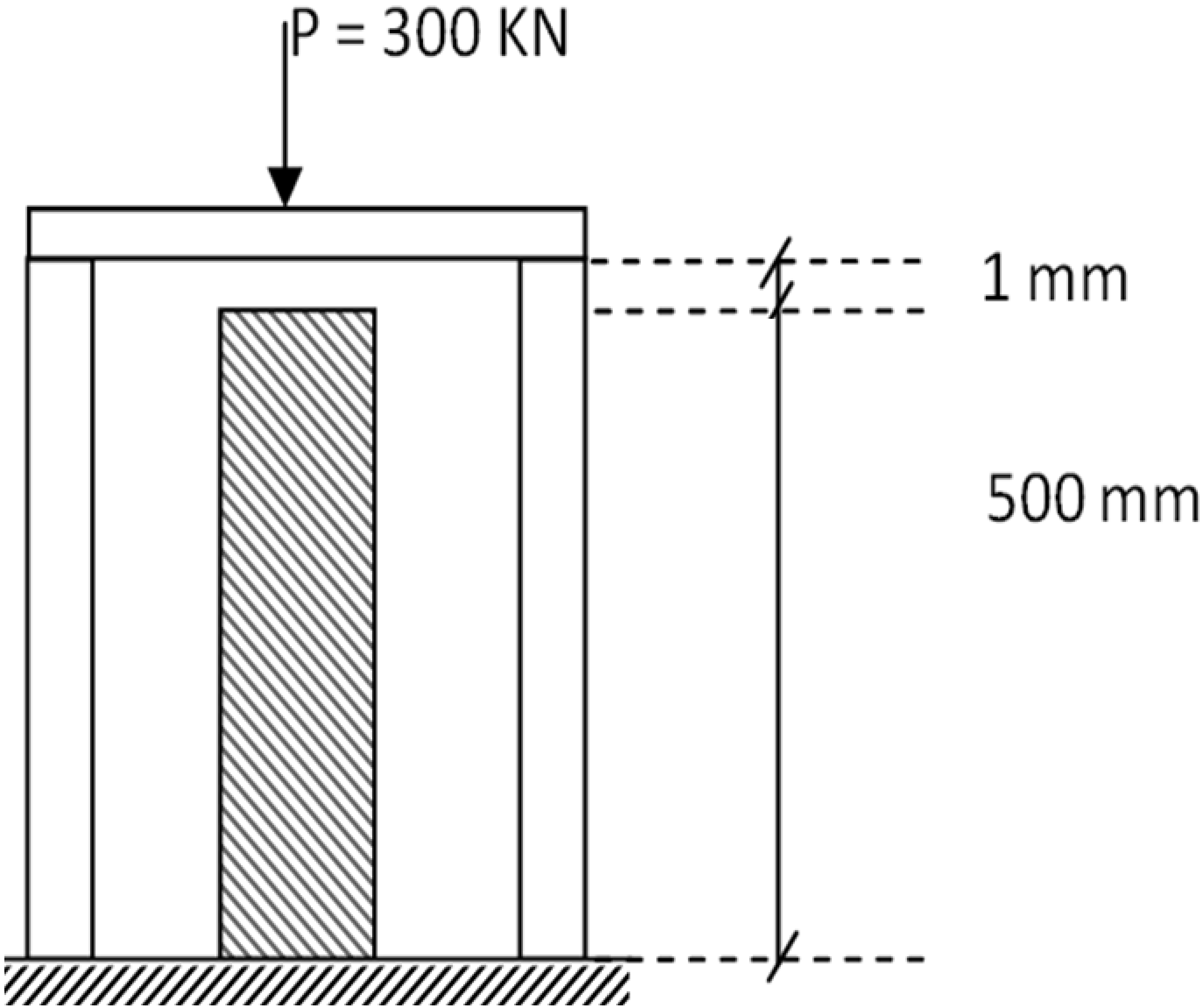
$$P_s = f_s A_s = 315.13 \text{ KN}$$

$$P_c = f_C A_C = 184.87 \text{ KN}$$

Ex 4:

A steel rod of diameter 30 mm and length 500 mm is placed inside a aluminium tube of internal diameter 35 mm and external diameter 45 mm which is 1 mm longer than the steel rod. A load of 300 kN is placed on the assembly through the rigid collar. Find the stress induced in steel rod and aluminium tube

Take E_s = 200 GPa and E_a = 80 GPa


P = 300 KN
1 mm
500 mm

Solution

$$A_s = \frac{\pi}{4} \times 30^2 = 706.86 \text{ mm}^2$$

$$A_a = \frac{\pi}{4} \times (45^2 - 35^2) = 628.32 \text{ mm}^2$$

Initial load carried by aluminium tube alone corresponding to a compression of 1 mm (P_a')

$$P_a' = \frac{A_a E_a \Delta}{l_a} = \frac{628.32 \times 80 \times 10^3 \times 1}{501}$$

$$= 100330 \text{ N}$$

Load for composite action,

$$P'' = 300000-100330$$

$$= 199670 \text{ N}$$

By equilibrium condition of composite action,

$$f_s'' A_s + f_a'' A_a = P''$$

$$708.86\, f_s'' + 628.32\, f_a'' = 199670 \qquad \ldots\ldots\ldots\ldots\ldots (1)$$

By compatibility condition

$$f_s'' = \frac{E_S}{E_c} \times f_a''$$

$$= \frac{200}{80} \times f_a''$$

$$f_s'' = 2.5\, f_a'' \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2)$$

Solving (1) and (2), we get

$$f_a'' = 83.353 \text{ N/ mm}^2$$

$$f_s'' = 208.383 \text{ N/ mm}^2$$

Final stress in steel $f_s = f_s' + f_s''$

$$=208.33\ \text{N/mm}^2$$

Final stress in aluminium $f_a = \frac{100330}{628.32} + 83.33$

$$= 159.68+83.353$$

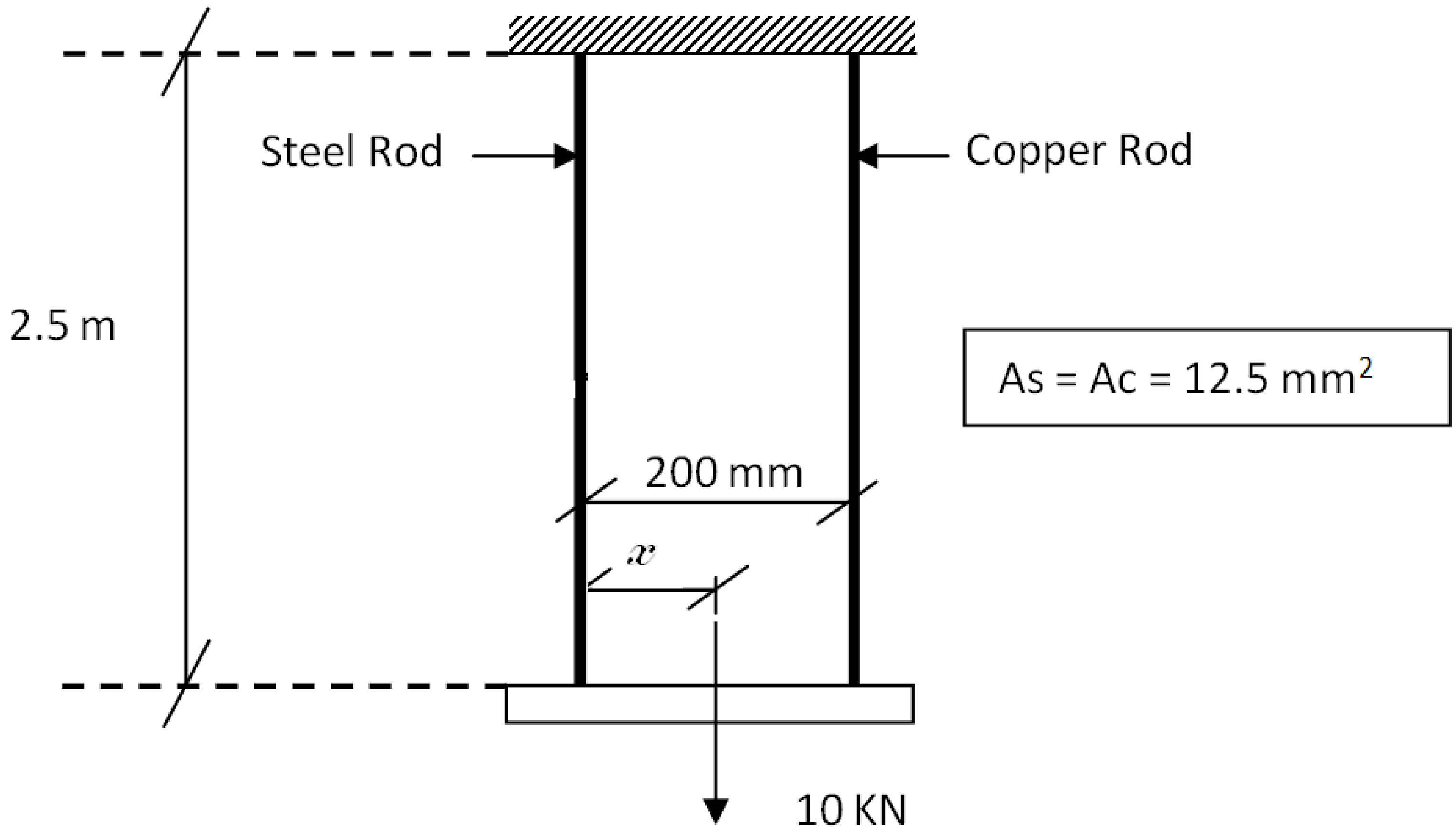
$$= 243.03\ \text{N/mm}^2$$

$$P_s = 147.3\ \text{KN}$$

$$P_a = 152.7\ \text{KN}$$

Ex 5

Two vertical rods one of steel and the other of copper are rigidly fastened at their upper end at a horizontal distance of 200 mm as shown in fig. the lower end supported a rigid horizontal bar, which carries a load of 10 kN. Both the rods are 2.5 m long and have cross sectional area of 12.5mm^2. Where should a load of 10 kN be placed on the bar, so that it remains horizontal after loading. Also find the stresses in each rod. Take E_s= 200 Gpa and E_c=110Gpa. Neglect bending of cross bar.


Steel Rod
Copper Rod
2.5 m
As = Ac = 12.5 mm²
200 mm
x
10 KN

by equilibrium condition,

$$f_s A_s + f_C A_C = P$$

$$12.5(f_s + f_C) = 10 \times 10^3$$

$$f_s + f_C = 800 \_\_\_\_\_\_ (1)$$

by compatibility, $\Delta_s = \Delta_c \quad e_s = e_c$

$$f_s = \frac{E_S}{E_c} \times f_c = \frac{200}{110} \times f_c$$

$$= 1.8182\, f_c \_\_\_\_\_\_ (2)$$

Solving (1) & (2), we get

$$f_c = 283.87 \text{ N}/\ mm^2$$

$$f_s = 516.13 \text{ N}/\ mm^2$$

$$P_s = 6451.64 \text{ N}$$

$$P_a = 3548.3 \text{ N}$$

Position of load

$$x = \frac{3.5483 \times 200}{10} = 71mm$$

Thermal stresses

Whenever there is some increase or decrease in the temperature of a body, it causes the body to expand or contract.

If the body is allowed to expand or contract freely, with the rise or fall of temperature, no stresses are induced in the body.

Thermal stresses ….

If the deformation of the body is prevented, some stresses are induced in the body. Such stresses are called thermal stresses or temperature stresses.

Change in length of a bar due to change in temperature, $\Delta = l\alpha t$

Where , l = length of the member

α = co efficient of thermal expansion of the material

t = change in temperature

Note: strain = α t

Thermal stresses

We know, for an axially loaded bar, $\Delta = \dfrac{pl}{AE}$

$$\text{Stress induced } f, = \frac{E\Delta}{l}$$

$$= \frac{El\alpha t}{l}$$

$$= E\alpha t$$

<u>Note</u>

'f' is compressive in nature due to increase in temperature and tensile in nature due to decrease in temperature

Ex:6

A steel rod of length 3m, diameter 20mm is subjected to an increase in temperature of 100°C. Find (i) the free expansion of the rod and stress induced in the rod when the expansion is fully prevented.
(ii) the stress in the rod when the extension permitted is 1mm.
Take $\alpha = 12\times10^{-6}/°c$ and $E_S = 200$ Gpa.

Solution

Free expansion of steel rod = lαt

$$= 3000 \times 12 \times 10^{-6} \times 100$$

$$= 3.6\ mm$$

Stress induced when the expansion is fully prevented ,f = Eαt

$$= 200 \times 10^{3} \times 12 \times 10^{-6} \times 100$$

$$= 240 N/mm^{2}$$

If the extension permitted, $\lambda = 1mm$

$\therefore$ extension prevented $= (l\alpha t - \lambda)$

$= 3.6 - 1$

$= 2.6mm$

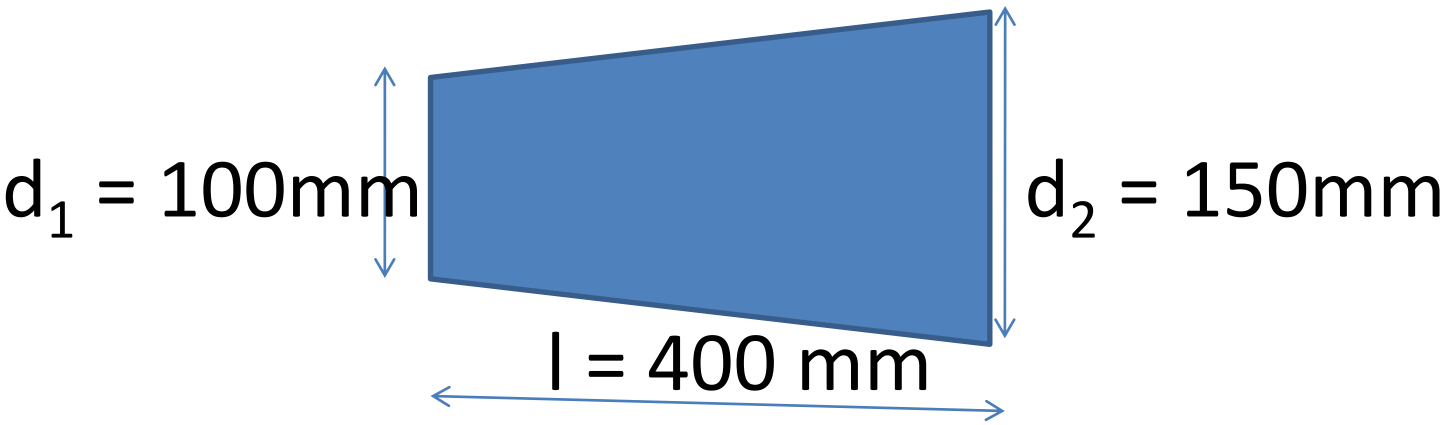
$$\text{stress induced(f)} = \frac{E}{l}(l\alpha t - \lambda)$$

$$= \frac{200 \times 10^3 \times 2.6}{3000}$$

$$= 173.33\ N/mm^2$$

Ex:7 A tapered bar as shown in Fig. below is subjected to a increase in temperature of 90˚C. Find maximum and minimum stress induced in material when expansion is totally prevented.

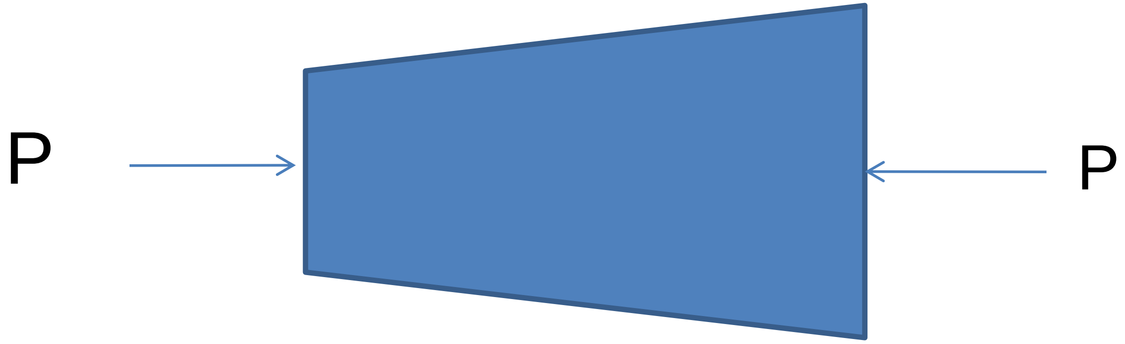
Take $\alpha = 12 \times 10^{-6}$ / ˚C, $E = 2 \times 10^{5}$ N/mm^2.



Soln:

Free expansion, $\Delta = l\,\alpha T$

If the expansion of the bar is totally prevented, then this is equivalent to the same bar is subjected to a compressive force (P)



Soln:

$$\Delta = \frac{4Pl}{\pi\, E d_1 d_2}$$

$$P = \frac{\pi E \Delta d_1 d_2}{4l} = 2544690\ N$$

$$Max\ stress = f_{max} = \frac{P}{A_{min}}$$

$$= \frac{2544690}{\frac{\pi}{4} \times 100^2}$$

$$= 323.99\ N/mm^2$$

$$(compressive)$$

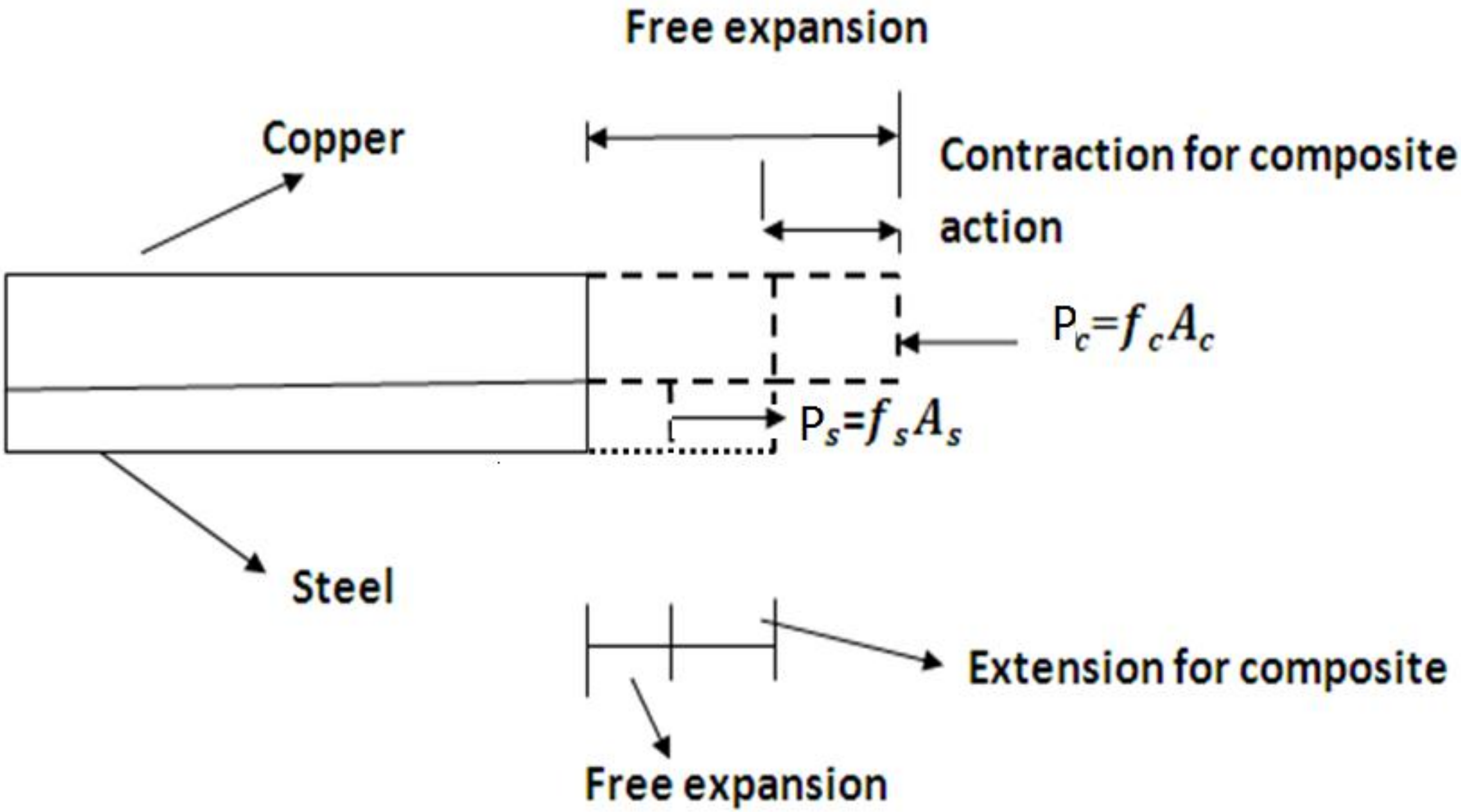
$$Min\ stress = f_{min} = \frac{P}{A_{max}}$$

$$= \frac{2544690}{\frac{\pi}{4} \times 150^2}$$

$$= 143.99\ N/mm^2$$

$$(compressive)$$

Thermal stress on composite bar $(\alpha_c > \alpha_s)$



Ex:8 A composite bar is made with a copper flat of size 50×30mm and steel flat of 50×40mm of length 500mm each placed one over the other. Find the stress induced in the material, when the composite bar is subjected to an increase in temperature of 100°C. Take

$$\alpha_s = 12 \times 10^{-6}/°\mathrm{c} \quad \alpha_c = 18 \times 10^{-6}/°\mathrm{c}$$

$$E_S = 2 \times 10^5\ \mathrm{N}/mm^2$$
$$E_C = 1 \times 10^5\ \mathrm{N}/mm^2$$

Free expansion of copper = $l\alpha_c t$

$$=500\times18\times10^{-6}\times100=0.9\text{ mm}$$

Free expansion of steel $=l\,\alpha_s t$

$$=500\times12\times10^{-6}\times100=0.6\text{ mm}$$

For composite action;

There is a push in copper and pull in steel.

For equilibrium,

push in copper (p_c) = pull in steel (p_s)

$$p_c = p_s$$

$$f_c A_c = f_s A_s$$

$$f_c = \frac{A_s}{A_c} \times f_s = 1.33 f_s$$

$$f_c = 1.33 f_s \quad \ldots\ldots\ldots\ldots \quad (1)$$

Actual expansion of copper (extension)

= (free expansion of copper –

contraction due to composite action)

$$= 0.9 - \frac{P_c\, l_c}{A_c E_c} \quad = 0.9 - \frac{f_c\, l_c}{E_c}$$

Actual expansion of steel

= (free expansion of steel) +

(expansion due to composite action)

$$= 0.6 + \frac{f_s\, l_s}{E_s}$$

For composite action,

$$0.9 - \frac{f_c\, l_c}{E_c} = 0.6 + \left(\frac{f_s\, l_s}{E_s}\right) \quad ....(2)$$

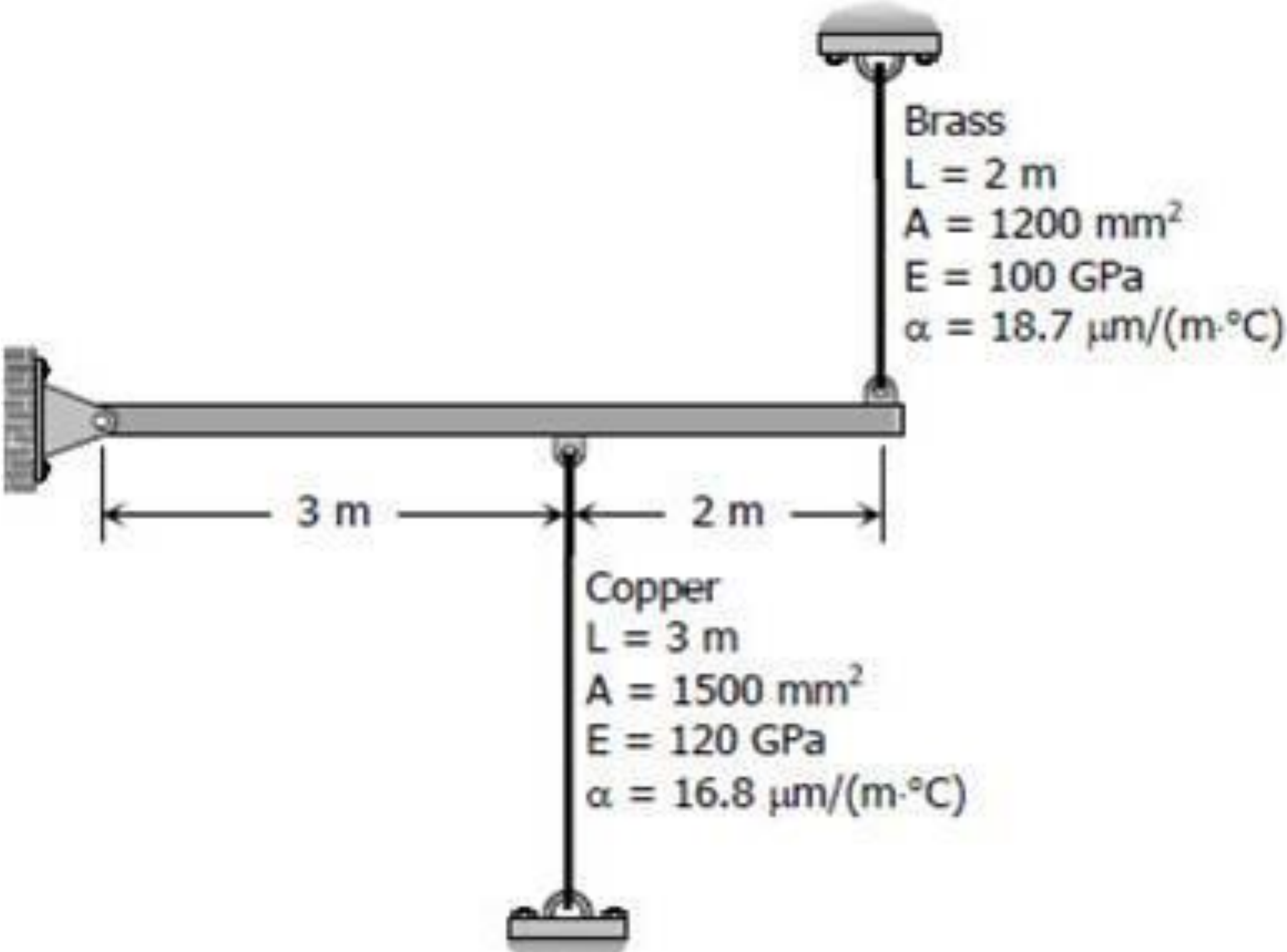
Sub (1) in (2)

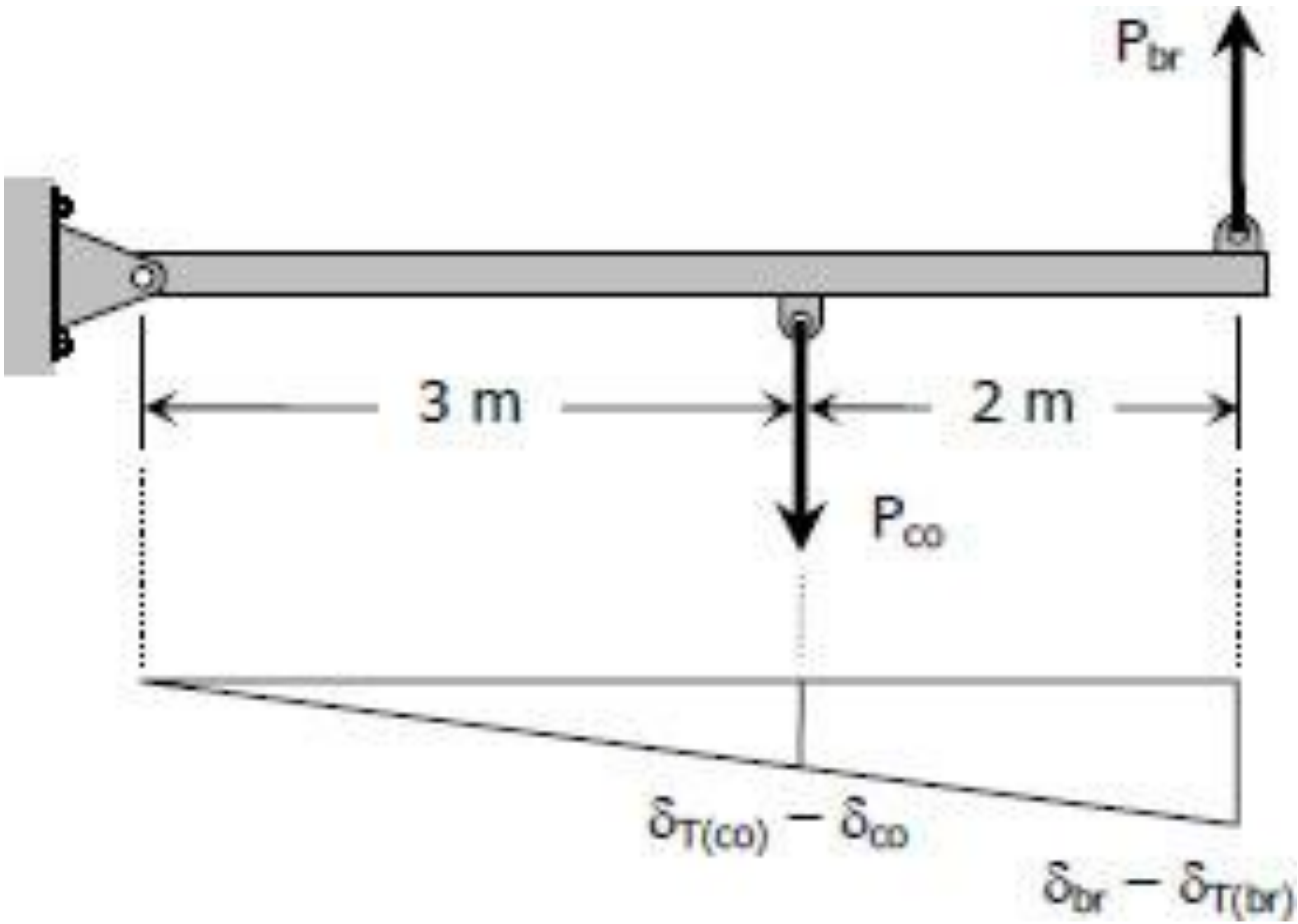
$$0.9 - \frac{1.33 f_{s} \times 500}{1 \times 10^5} = 0.6 + \frac{f_{s} \times 500}{2 \times 10^5}$$

$f_s = 32.78\ N/mm^2$ (T)

$f_c = 43.60\ N/mm^2$ (C)

Ex:9 A rigid horizontal bar of negligible mass is connected to two rods as shown in Fig. If the system is initially stress-free. Calculate the temperature change that will cause a tensile stress of 90 MPa in the brass rod. Assume that both rods are subjected to the change in temperature.


Brass
L = 2 m
A = 1200 mm²
E = 100 GPa
α = 18.7 μm/(m·°C)
3 m
2 m
Copper
L = 3 m
A = 1500 mm²
E = 120 GPa
α = 16.8 μm/(m·°C)



P_{br}
3 m
2 m
P_{co}
$\delta_{T(co)} - \delta_{co}$
$\delta_{br} - \delta_{T(br)}$

Solution

Let , P_{br} be the force in brass rod and P_{co} be the force in brass rod due to the change in temperature.

$$\sum M_{hinge\ support} = 0$$

$$5P_{br} - 3P_{co} = 0$$

$$5\sigma_{br}A_{br} - 3\sigma_{co}A_{co} = 0$$

$$5(90)(1200) - 3\sigma_{co}(1500) = 0$$

$$\sigma_{co} = 120\ MPa$$

(Stress induced in the copper rod)

We know, $$\delta = \frac{\sigma L}{E}$$

$$\delta_{br} = \frac{90\ (2000)}{100000} = 1.8\ mm$$

$$\delta_{co} = \frac{120\ (3000)}{120000} = 3\ mm$$

Where δ_{br} and δ_{co} are the amount of deformations prevented in the brass and copper rod respectively

Let $\delta_{T(br)}$ and $\delta_{T(co)}$ are the amount of free deformations induced due to change in temperature in the brass and copper rod respectively.

Net deformation of brass rod

$$= \delta_{br} - \delta_{T(br)}$$ and

net deformation of copper rod

$$= \delta_{T(co)} - \delta_{co}$$

$$\frac{\delta_{T(co)} - \delta_{co}}{3} = \frac{\delta_{br} - \delta_{T(br)}}{5}$$

$$5\delta_{T(co)} - 5\delta_{co} = 3\delta_{br} - 3\delta_{T(br)}$$

$$5\,(3000)(16.8 \times 10^{-6})\,\Delta T - 5(3)$$
$$= 3(1.8) - 3(2000)(18.7 \times 10^{-6})\Delta T$$

$$0.3642\,\Delta T = 20.4$$

$$\Delta T = 56.01°\mathrm{C}$$

(drop in temperature)

Shrinking on stresses

The inner diameter of the steel tyre is slightly greater than the outer diameter of the wooden wheel.

The steel tyre is heated and slipped on to the wooden wheel and cooled to fit.

Strain in the steel tyre

$= (\pi D - \pi d)/ \pi d = (D - d)/d$

Stress in the steel tyre

$= E (D - d)/d$

ELASTIC CONSTANTS

Poisson's ratio (***μ***), is the ratio, when a sample object is stretched, of the contraction or **transverse strain** (perpendicular to the applied load), to the extension or **axial strain** (in the direction of the applied load).

i.e., it is the ratio of lateral strain to longitudinal strain.

When a material is compressed in one direction, it usually tends to expand in the other two directions perpendicular to the direction of compression. This phenomenon is called the **Poisson effect**.

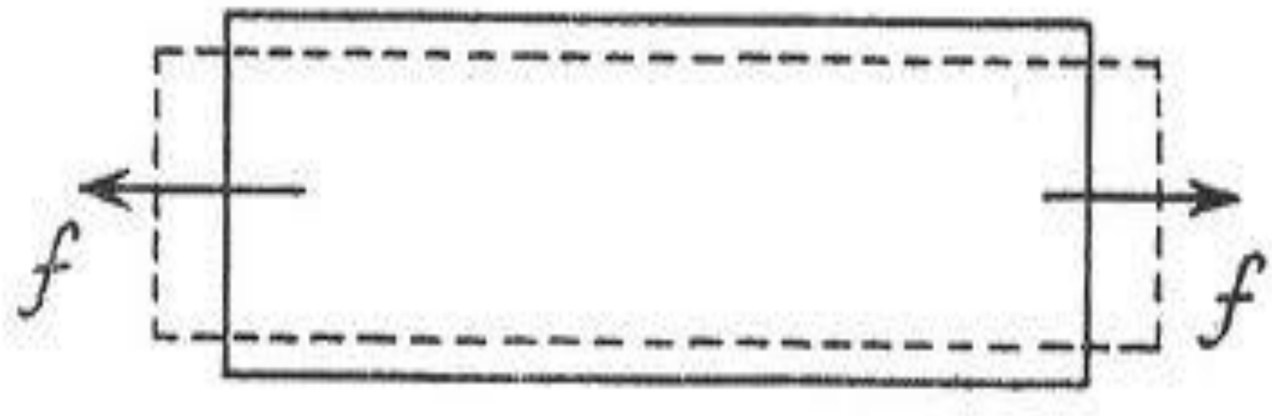
Poisson's ratio μ(nu) is a measure of the Poisson effect.

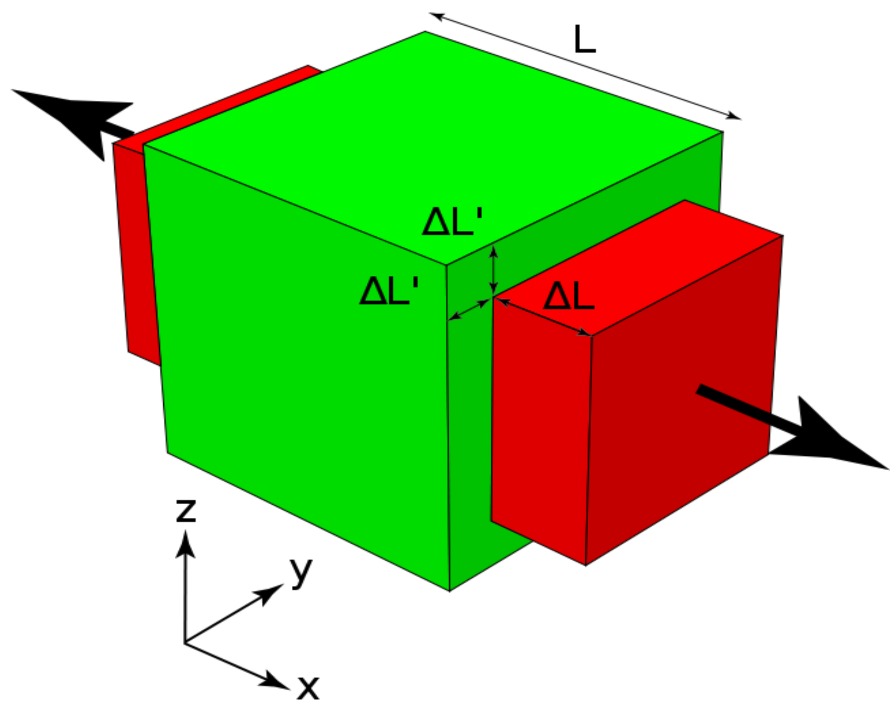
Conversely, if the material is stretched rather than compressed, it usually tends to contract in the directions transverse to the direction of stretching. Again, the Poisson ratio will be the ratio of relative contraction to relative stretching, and will have the same value as above.

- Most materials have Poisson's ratio values ranging between 0.0 and 0.5.
- A perfectly incompressible material deformed elastically at small strains would have a Poisson's ratio of exactly 0.5.
- Rubber has a Poisson ratio of nearly 0.5.

- Most steels and rigid polymers when used within their design limits (before yield) exhibit values of about 0.3, increasing to 0.5 for post-yield deformation (which occurs largely at constant volume.)
- Cork's Poisson ratio is close to 0: showing very little lateral expansion when compressed

Materials	ν
Cork	~ 0
Concrete	0.1
Ceramics	0.2
Steel	0.3
Aluminum	0.33
Rubber	~ 0.5

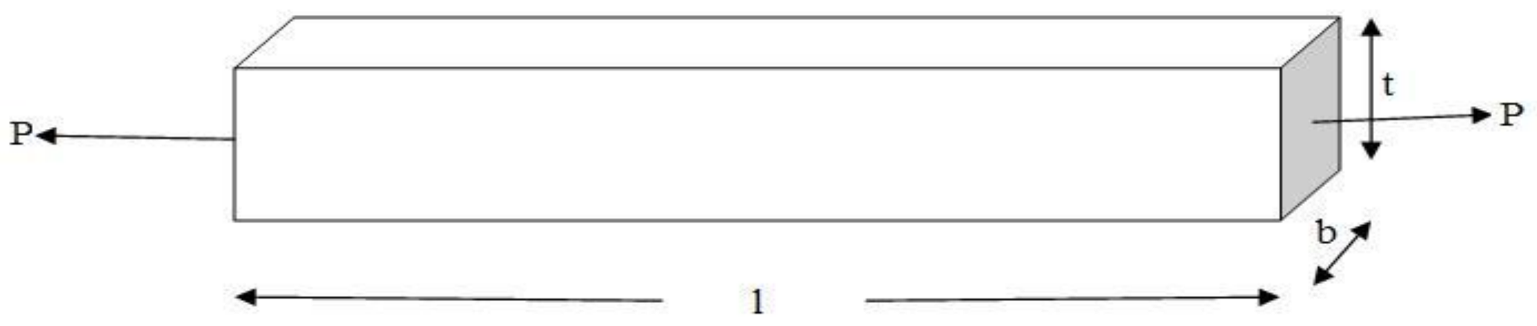

f
f


L
ΔL'
ΔL'
ΔL
z
y
x

Poisson's ratio,

$$\mu = \frac{lateral\ strain\ or\ transverse\ strain}{longitudinal\ strain}$$

Volumetric strain of a rectangular body subjected to an axial force



Initial volume, $V = lbt$

Final length, $l = l + \Delta l$

Final breadth, $b = b - \Delta b$

Final thickness, $t = t - \Delta t$

Final volume, $\overline{V} = (l + \Delta l)(b - \Delta b)(t - \Delta t)$

$$= lbt + \Delta lbt - lt\Delta b - \Delta l\Delta bt - lb\Delta t$$
$$- \Delta l\Delta tb + l\Delta b\Delta t + \Delta l\Delta b\Delta t$$

Neglecting the product of two or more small quantities

$$\bar{V} = lbt + bt\Delta l - lt\Delta b - lb\Delta t$$

Change in volume,

$$\Delta V = bt\Delta l - lt\Delta b - lb\Delta t$$

Volumetric strain,

$$e_v = \frac{change\ in\ volume}{initial\ volume} = \frac{\Delta V}{V}$$

$$= \frac{bt\Delta l - lt\Delta b - lb\Delta t}{lbt}$$

$$= \frac{\Delta l}{l} - \frac{\Delta b}{b} - \frac{\Delta t}{t}$$

$$= e_x + e_y + e_z$$

(or)

$$V = lbt$$

$$logV = logl + logb + logt$$

Differentiating, we get

$$\frac{dV}{V} = \frac{\Delta l}{l} + \frac{\Delta b}{b} + \frac{\Delta t}{t}$$

ie., $$e_V = (e_x + e_y + e_z)$$

Volumetric strain = Sum of linear strains

Note:

1) For a solid cylindrical bar of diameter ‘d’ and length ‘l’

$$V = \frac{\pi d^2}{4} l$$

$$\Delta V = \frac{\pi}{4}[d^2 \Delta l + 2dl.\Delta d]$$

$$e_V = \frac{\Delta V}{V} = \frac{\Delta l}{l} + 2\frac{\Delta d}{d} \;\; (= e_x + e_y + e_z)$$

Note:

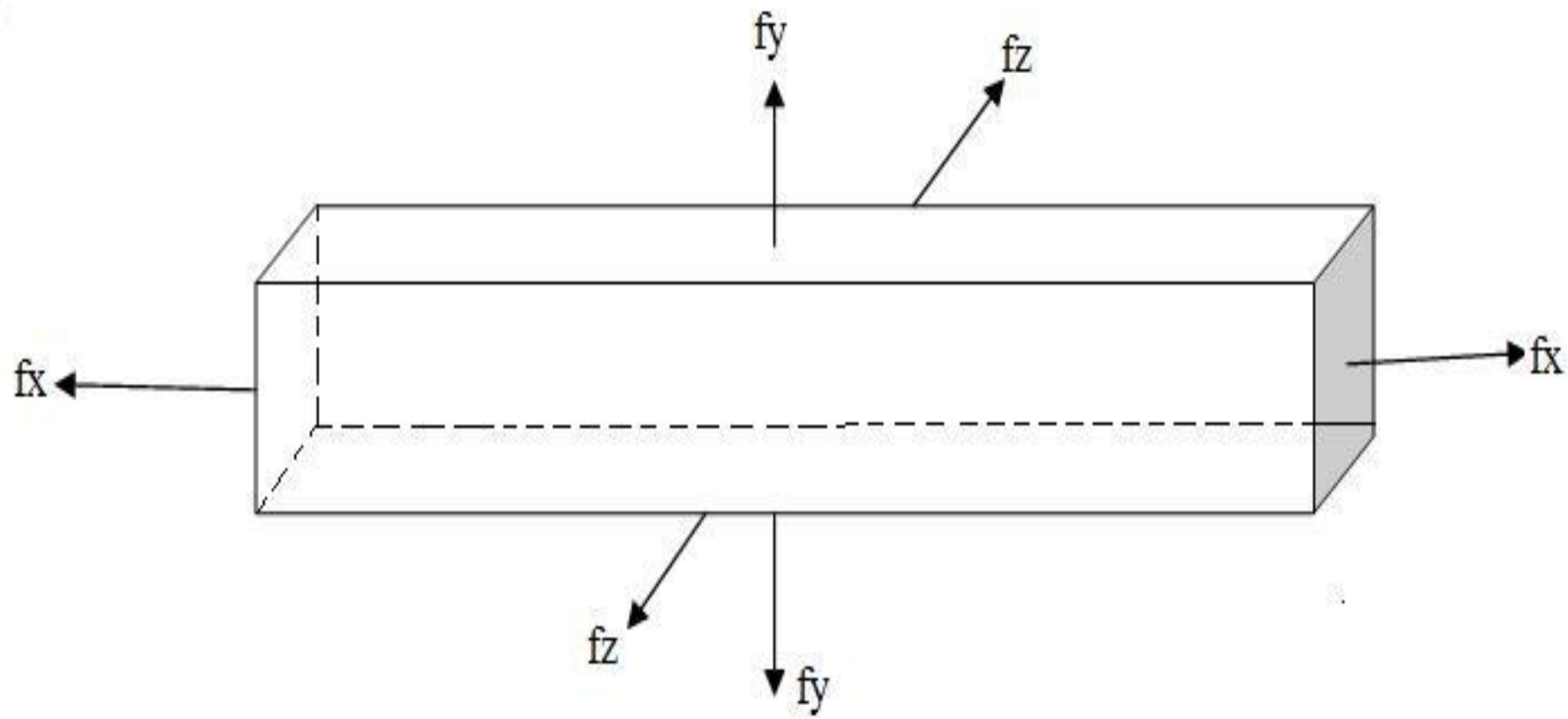
2) For a sphere of diameter 'D'

$$V = \frac{\pi D^3}{6}$$

$$\Delta V = \frac{\pi}{6}(3D^2 \Delta D)$$

$$e_V = 3\frac{\Delta D}{D}(= e_x + e_y + e_z)$$

Volumetric strain of a rectangular body subjected to three mutually perpendicular forces



The resultant strain in x direction,

$$e_x = \frac{f_x}{E} - \mu\frac{f_y}{E} - \mu\frac{f_z}{E}$$

Similarly

$$e_y = \frac{f_y}{E} - \mu\frac{f_x}{E} - \mu\frac{f_z}{E}$$

and

$$e_z = \frac{f_z}{E} - \mu\frac{f_x}{E} - \mu\frac{f_y}{E}$$

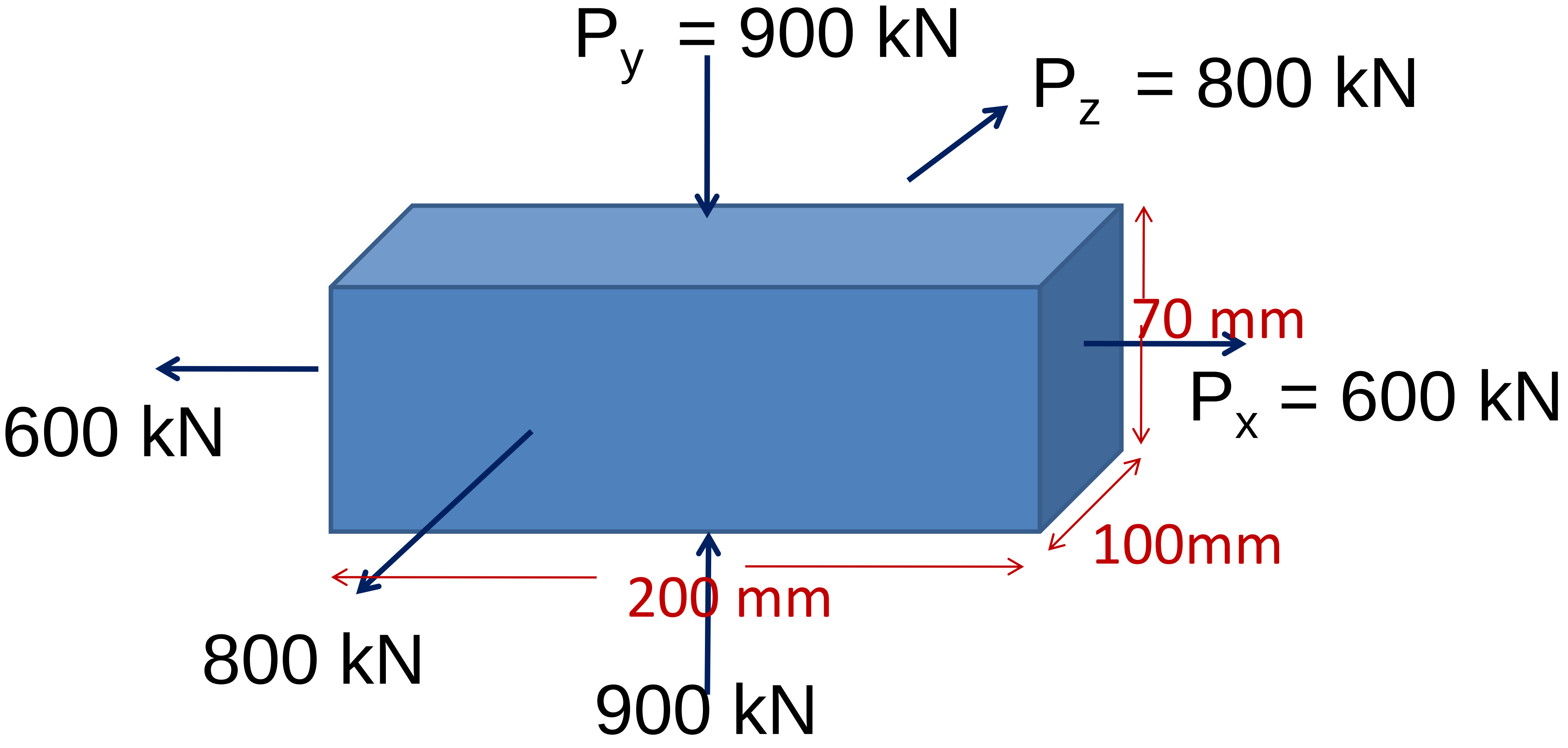
Volumetric strain, $e_V = e_x + e_y + e_z$

$$= \frac{f_x}{E}(1-2\mu) + \frac{f_y}{E}(1-2\mu)$$

$$+ \quad \frac{f_z}{E}(1-2\mu)$$

$$= \frac{(f_x + f_y + f_z)}{E}(1-2\mu)$$

Ex:10 A solid metallic bar of size 200mm × 100mm × 70mm is subjected to forces in three mutually perpendicular directions as shown in Fig. Find the change in volume of the bar. Take E=200GPa and Poisson's ratio, μ=0.25. Also find the change that should be made inthe 900kN load, in order that there should be no change in volume of the bar.


P_y = 900 kN
P_z = 800 kN
70 mm
P_x = 600 kN
600 kN
100mm
200 mm
800 kN
900 kN

Soln:

$$f_x = \frac{P_x}{A_x} = \frac{600 \times 10^3}{100 \times 70} = 85.7\ N/mm^2$$

$$f_y = \frac{P_y}{A_y} = \frac{-900 \times 10^3}{200 \times 100} = -45\ N/mm^2$$

$$f_z = \frac{P_z}{A_z} = \frac{800 \times 10^3}{200 \times 70} = 57.14\ N/mm^2$$

$$\therefore\ e_v = \left(f_x + f_y + f_z\right)\frac{(1-2\mu)}{E}$$

$$= 2.45\ \times 10^{-4}$$

$\therefore$ change in volume,

$$\Delta V = e_v = 2.45 \times 10^{-4} \times 200 \times 100 \times 70$$

$$= 342.4\ mm^3$$

Change required in the 900 kN load for no change in volume

Let P be the compressive load in kN that should be made to act in the plane to 900 kN load in order that there should be no change in volume of the bar (no change in volumetric strain)

For, $e_v = 0$

$$\left(f_x + f_y + f_z\right)\frac{(1-2\mu)}{E} = 0$$

$$f_y = -f_x - f_z$$

$$\frac{-P}{200 \times 100} = -85.7 - 57.14$$

$$\therefore P = 2856.8\ kN$$

Bulk modulus

When a body is subjected to three mutually perpendicular stresses of equal intensity, the ratio of direct stress to the corresponding volumetric strain is called Bulk modulus (K)

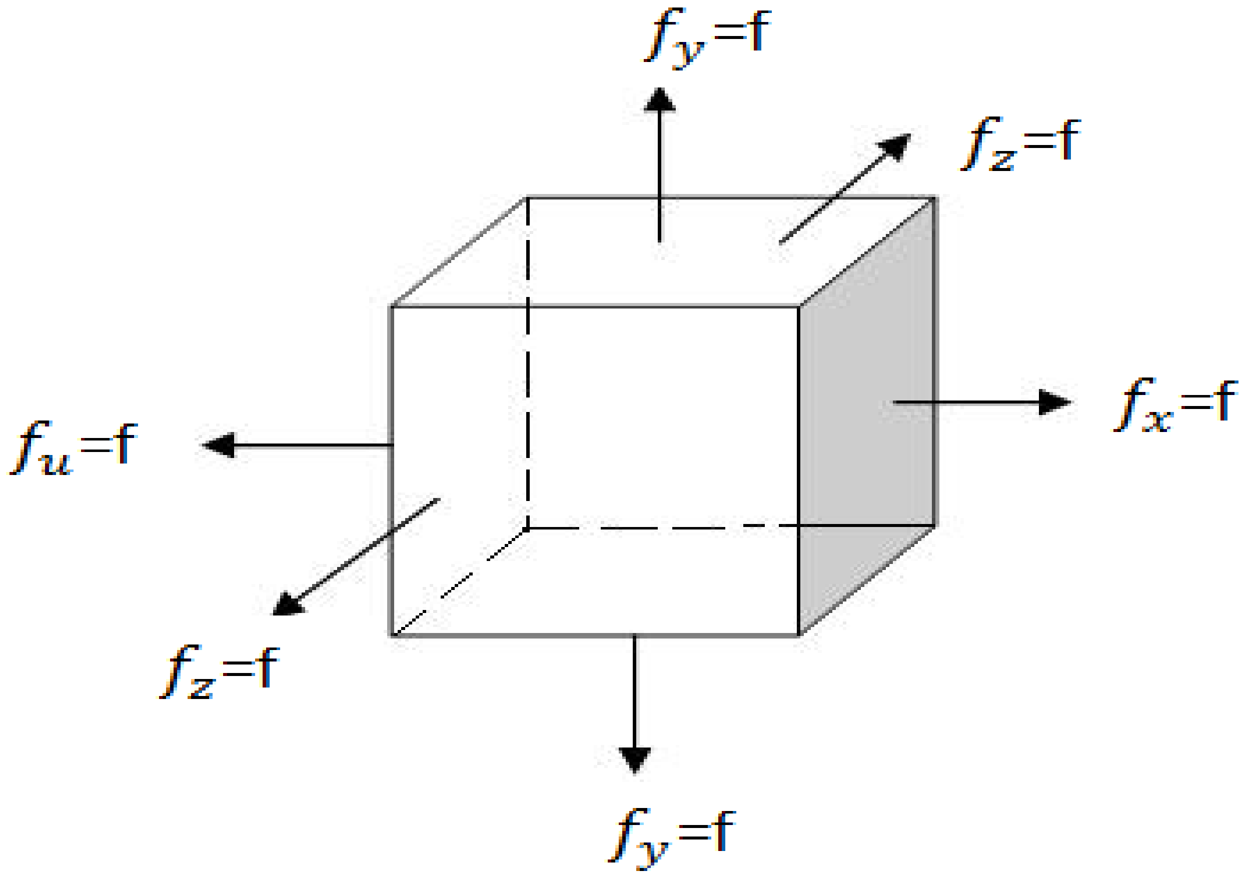
$$K = \frac{direct\ stress}{volumetric\ strain}$$

Bulk modulus

The ratio of the stress on the body to the body's fractional decrease in volume is the **bulk modulus.**

i.e., it is defined as the direct stress required to produce unit volumetric strain in the body.

Relation between Bulk Modulus and Modulus of Elasticity



Volumetric strain,

$$e_V = (f_x + f_y + f_z)\frac{(1-2\mu)}{E}$$

$$= \frac{3f}{E}(1-2\mu)$$

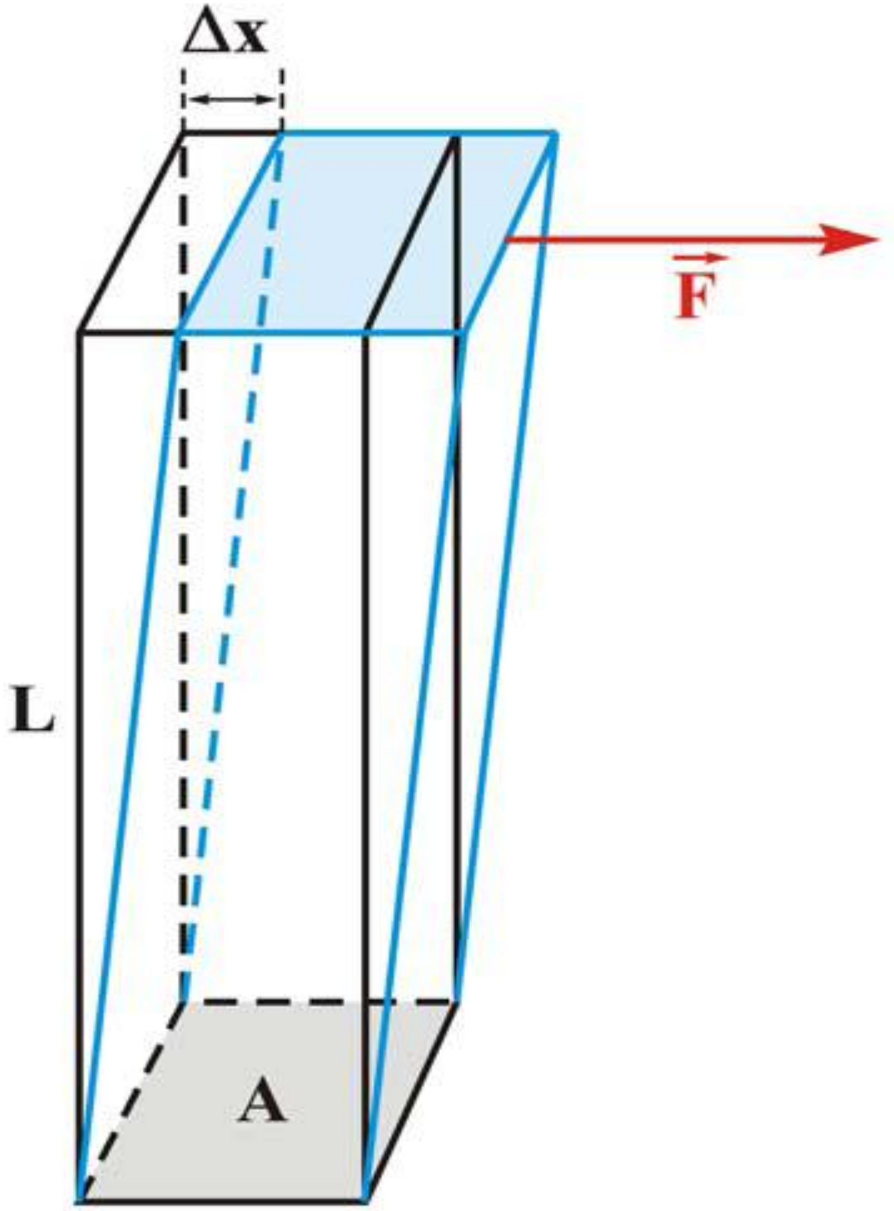
$$E = \frac{3f}{e_v}(1-2\mu)$$

$$E = 3K(1-2\mu\ )$$

Shear modulus

- Shear Modulus or Modulus of Rigidity is the coefficient of elasticity for a shearing or torsion force.
- The ratio of the tangential force per unit area to the angular deformation in radians is the **shear modulus.**

- The shear modulus is the elastic modulus we use for the deformation which takes place when a force is applied parallel to one face of the object while the opposite face is held fixed by another equal force.


Δx
$\vec{F}$
L
A

- When an object like a block of height L and cross section A experiences a force F parallel to one face, the sheared face will move a distance Δx. The shear stress is defined as the magnitude of the force per unit cross-sectional area of the face being sheared (F/A). The shear strain is defined as Δx/L.

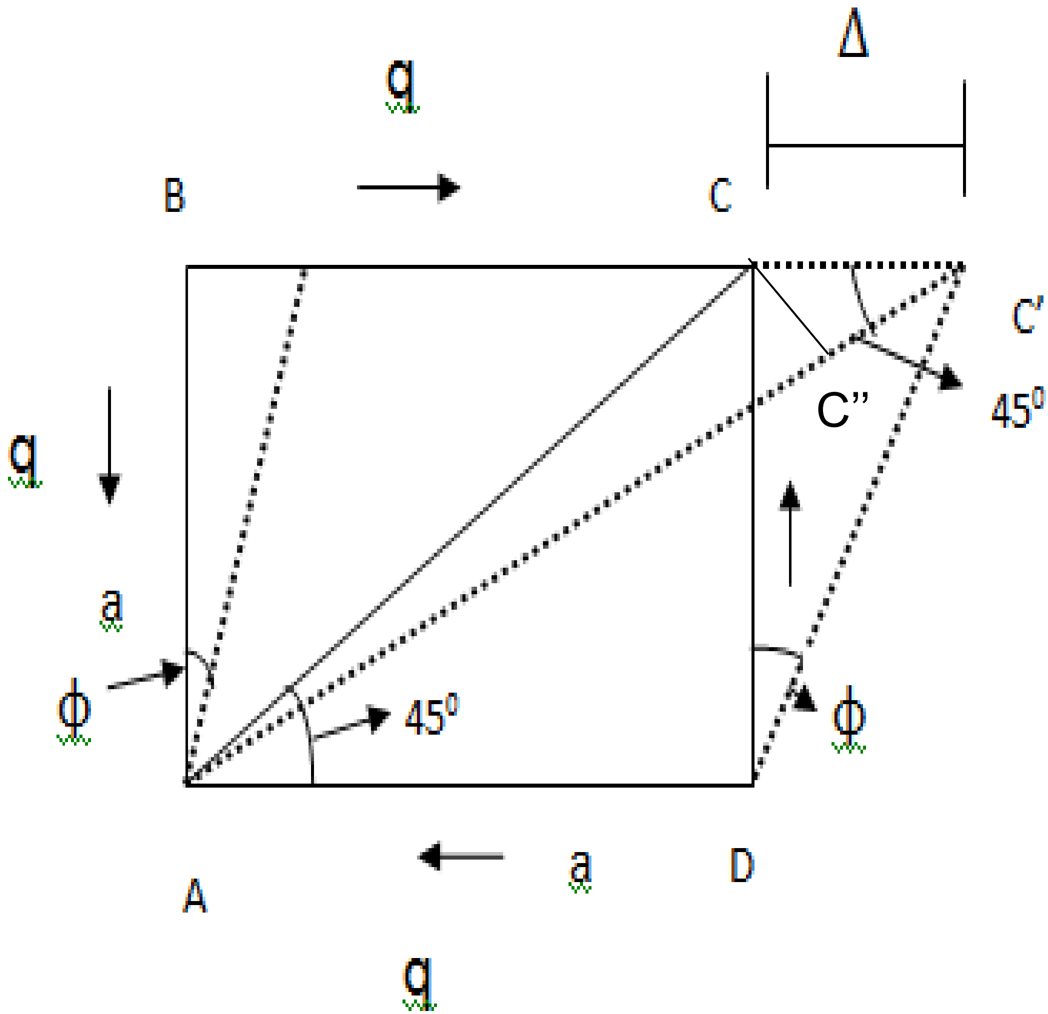
The shear modulus N is defined as the ratio of the shearing stress to the shearing strain.

Shear modulus,

$$N = \frac{shear\ stress}{shearing\ strain}$$

$$= \frac{q}{\emptyset}$$

Relation between Shear Modulus and Modulus of Elasticity



Shearing strain $\phi = \dfrac{\Delta}{a}$

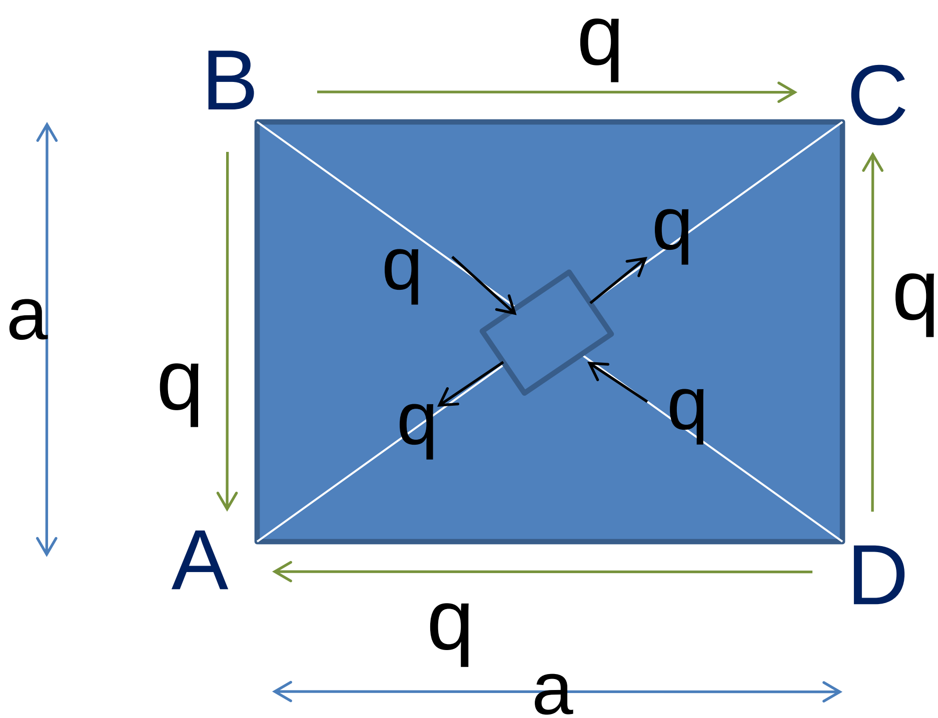
Strain in the diagonal AC $= \dfrac{c'c''}{AC}$

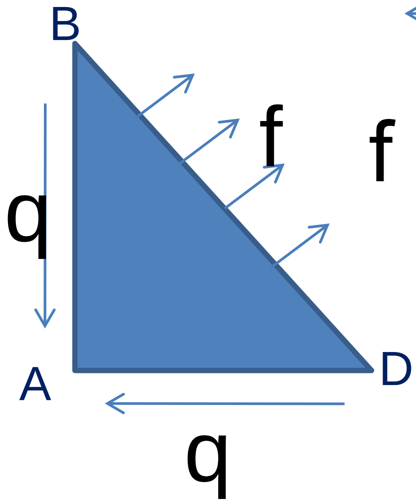
$$= \frac{\frac{\Delta}{\sqrt{2}}}{a\sqrt{2}} = \frac{\Delta}{2a}$$

$$= \frac{\phi}{2} \quad \text{\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_}(1)$$

Linear strain in the diagonal AC

$$= \frac{1}{2} \times shearing\ strain$$





$$f \cos 45 \,(a\sqrt{2} * 1) - q\,(a * 1) = 0$$

$$f = q$$

Considering an element in the diagonal,

Strain in the diagonal AC,

$$= \frac{q}{E} - \mu(-\frac{q}{E})$$

$$= \frac{q}{E}(1+\mu) \text{ \_\_\_\_\_\_\_\_\_\_\_\_ } (2)$$

From (1) & (2) $\quad \frac{\emptyset}{2} = \frac{q}{E}(1+\mu)$

$$E = \frac{2q}{\emptyset}(1+\mu)$$

$$E = 2N(1+\mu)$$

Relation between E,K and N

We know that,

$$E = 2N(1 + \mu)\_\_\_\_\_\_\_\_\_(1)$$

$$E = 3K(1 - 2\mu)\_\_\_\_\_\_\_\_\_(2)$$

From (2),

$$\mu = \frac{1}{2}(1 - \frac{E}{3K})\_\_\_\_\_\_\_\_\_(a)$$

From (1),

$$\mu = \frac{E}{2N} - 1 \_\_\_\_\_\_\_\_\_\_\_\_ (b)$$

$$\therefore \frac{1}{2}\left(1 - \frac{E}{3K}\right) = \frac{E}{2N} - 1$$

$$\frac{1}{2} - \frac{E}{6K} = \frac{E}{2N} - 1$$

$$\frac{3K - E}{6K} = \frac{E - 2N}{2N}$$

$$6KN - 2EN = 6KE - 12KN$$

$$18KN = 6KE + 2EN$$

$$= 2E(3K + N)$$

$$E = \frac{9KN}{3K + N}$$

Ex 11: A bar (square) of size 10mm × 10mm is subjected to an axial pull of 25kN. The measured extension over a gauge length of 200mm is 0.25mm. Final dimension of the bar is 9.997 × 9.997 mm. Find the Poisson's ratio μ and the three elastic modulii.

Soln: $L = 200\text{mm}$

$\Delta L = 0.25\text{mm}$

$$\mu = \frac{\text{lateral strain}}{\text{longitudinal strain}}$$

Soln:

$$\mu = \frac{3 \times 10^{-3}/10}{0.25/200} = 0.24$$

P = 25 kN

Δ = 0.25 mm

$$f = \frac{25 \times 10^{3}}{100} = 250\ N/mm^{2}$$

$$e = \frac{0.25}{200} = 1.25 \times 10^{-3}$$

$$E = \frac{P\,l}{A\,\Delta} = \frac{25 \times 10^3 \times 200}{100 \times 0.25}$$

$$= 2 \times 10^5\, N/mm^2$$

$$E = 3K\,(1 - 2\mu)$$

$$K = \frac{E}{(1 - 2\mu)} = 128205.13$$

$$= 1.28 \times 10^5 N/mm^2$$

$$E = 2N\,(1 + \mu)$$

$$N = \frac{E}{2(1 + \mu)}$$

$$= 0.8\ \times 10^5 N/mm^2$$

$$= 0.8\ \times 10^5 MPa$$

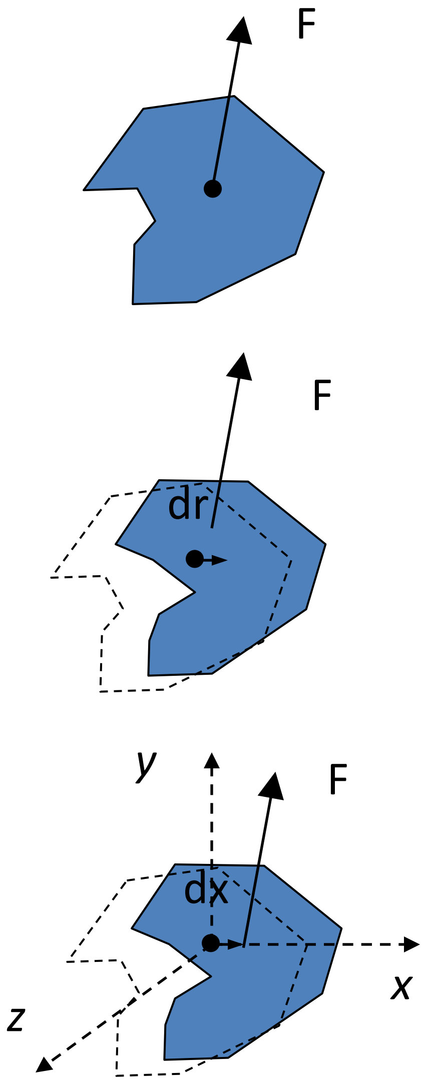
Strain energy

Whenever a body is strained, some amount of energy is absorbed in the body. The energy that is absorbed in the body due to straining effect is known as strain energy.

(or)

The potential energy stored in a body by virtue of an elastic deformation, equal to the work that must be done to produce this deformation.

Work and Energy



Consider a solid object acted upon by force, F, at a point, O, as shown in the figure.

The work done = F dr

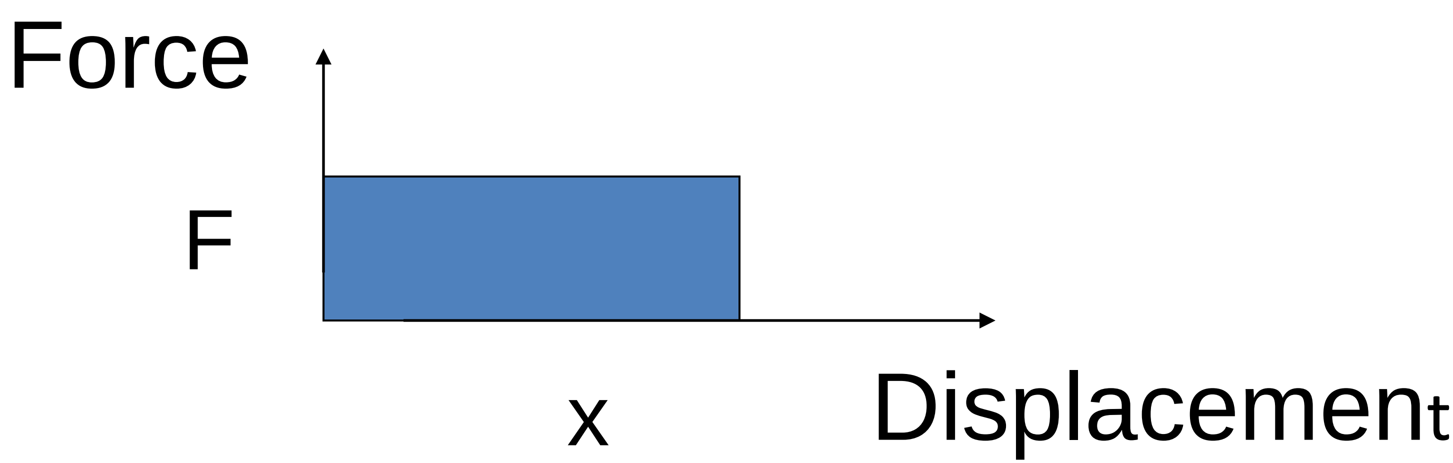
For the general case:

$W = F_x \, dx$

i.e., only the force in the direction of the deformation does work.

Amount of Work done

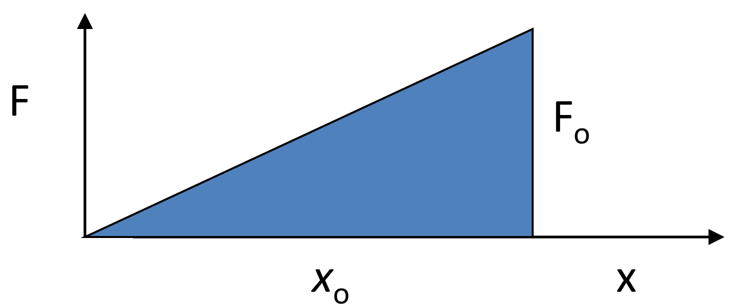
Constant Force: If the Force is constant, the work is simply the product of the force and the displacement, W = F X



Amount of Work done

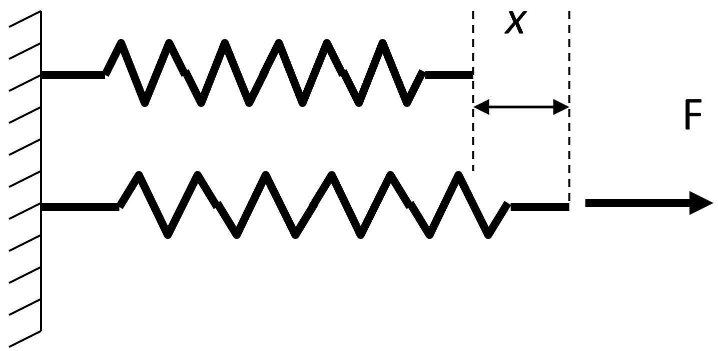
Linear Force: If the force is proportional to the displacement, the work is

$$W = \frac{1}{2} F_o x_o$$



Displacement

Strain Energy



Consider a simple spring system, subjected to a Force such that F is proportional to displacement *x*; $F=kx$.

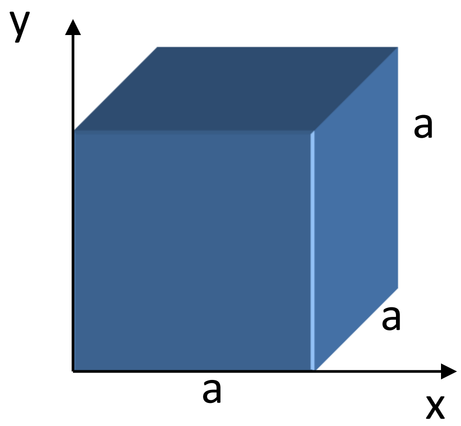
Strain energy $= kx^2$

Strain Energy

$$W = \frac{1}{2} F_o x_o$$

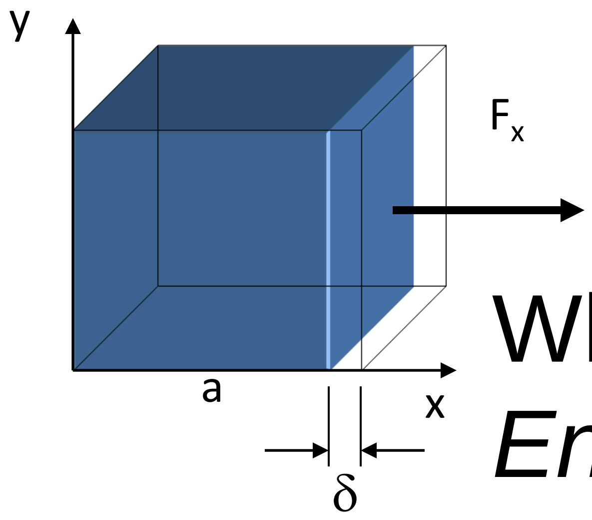
This energy (work) is stored in the spring and is released when the force is returned to zero

Strain Energy Density



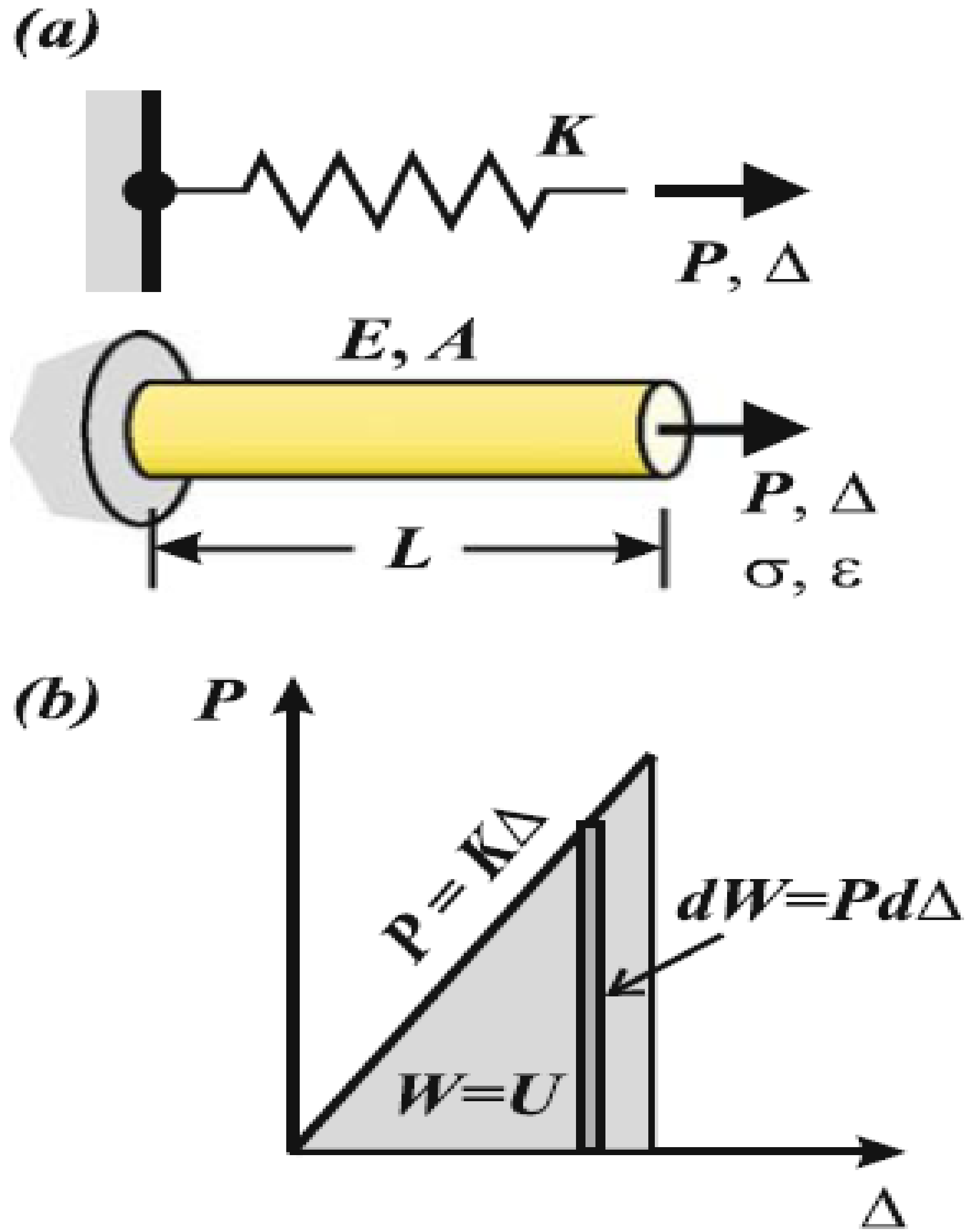
$$W = \frac{1}{2} F_x \delta$$

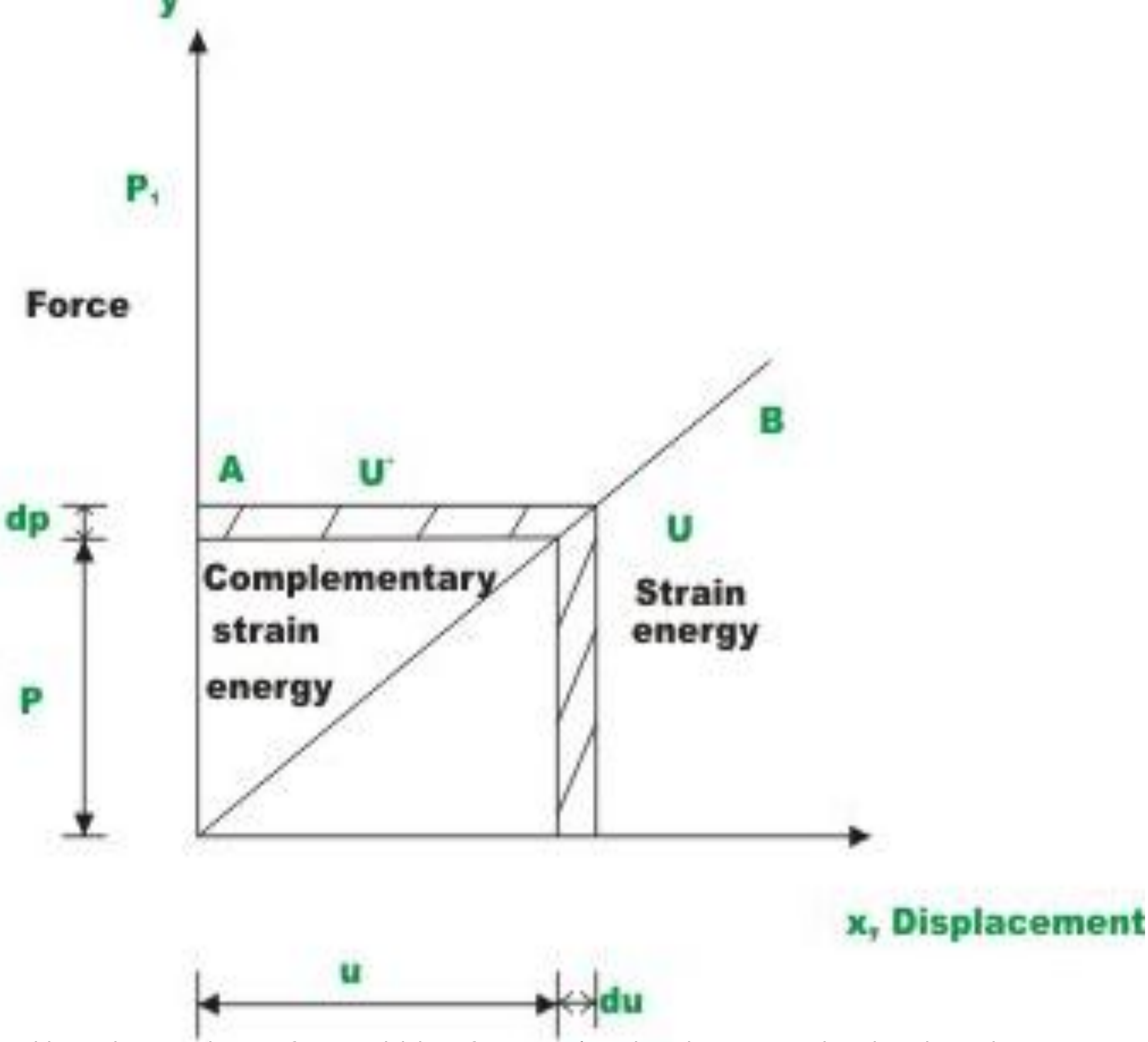
$$U = \frac{1}{2}\sigma_x a^2 e_x a = \frac{1}{2}\sigma_x e_x a^3$$



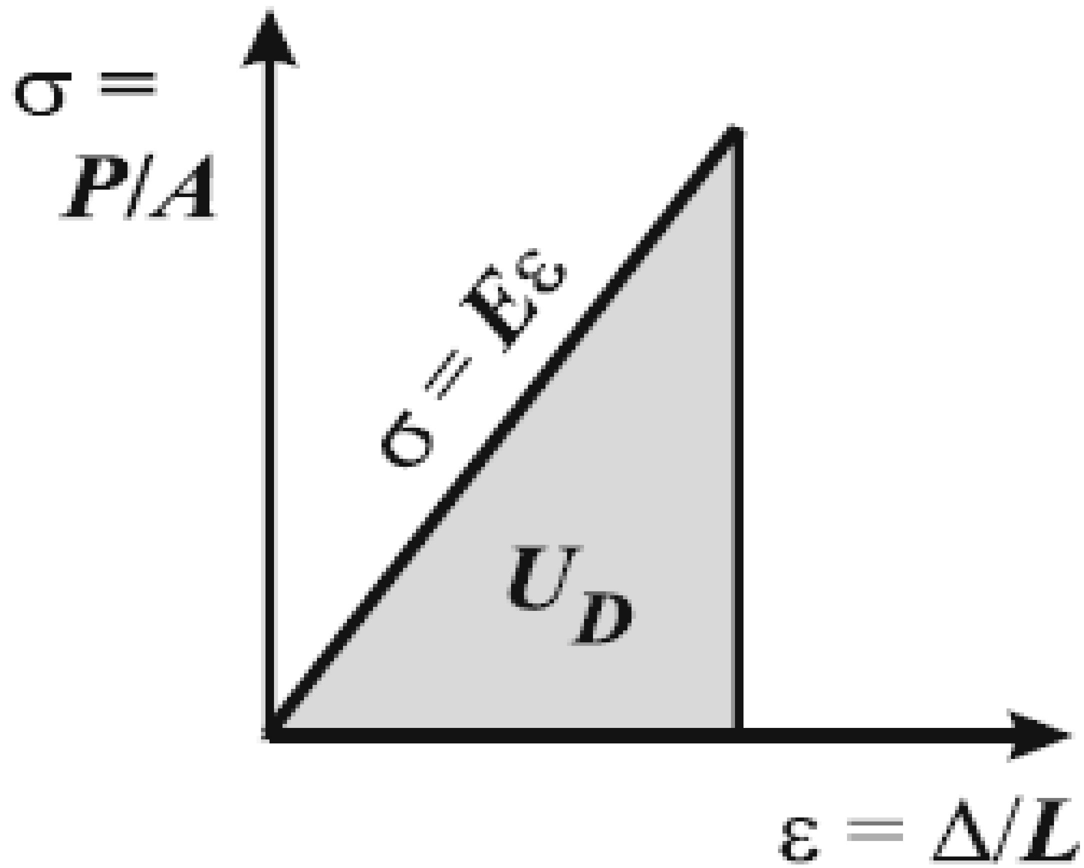
$$u = \frac{U}{V} = \frac{1}{2}\sigma_x e_x a^3 / a^3 = \frac{1}{2}\sigma_x e_x$$

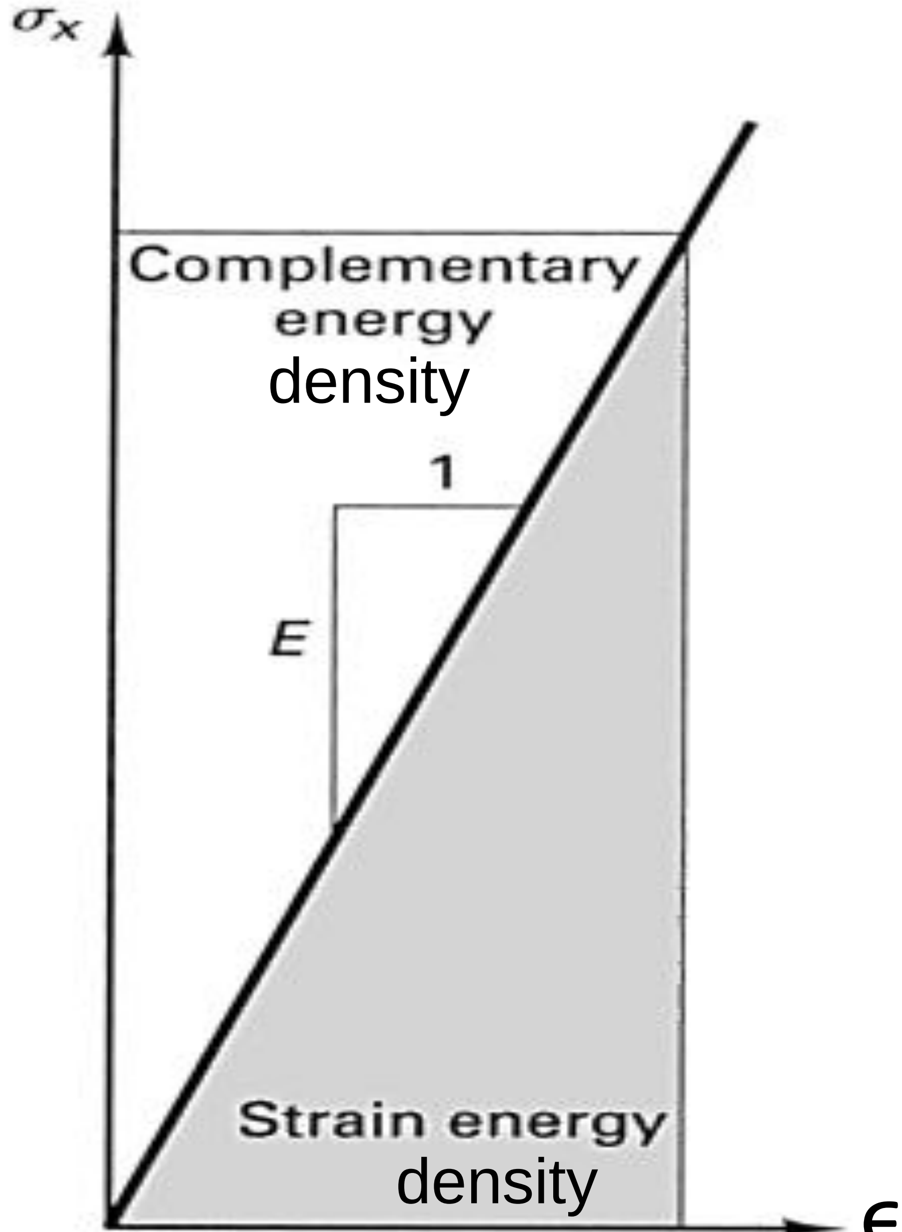
Where U is called the *Strain Energy*, and u is the *Strain Energy Density.*


(a)
K
P, Δ
E, A
P, Δ
L
σ, ε
(b)
P
P = KΔ
dW=PdΔ
W=U
Δ


y
P₁
Force
B
A
U*
dp
U
Complementary
strain
energy
Strain
energy
P
x, Displacement
u
du

Strain energy density



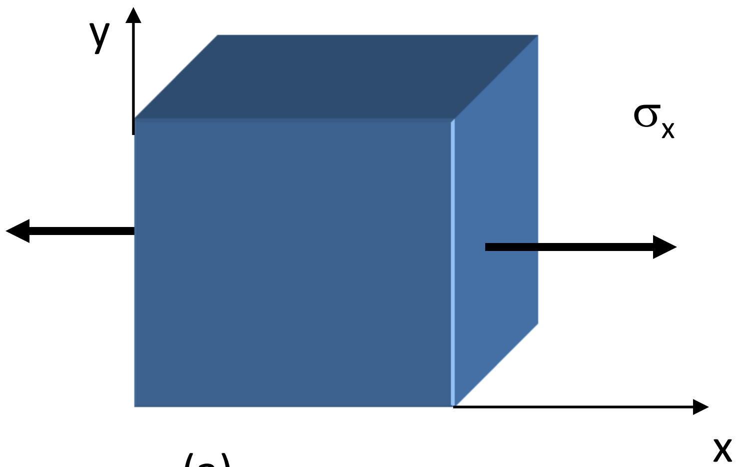

σ_x
Complementary energy density
1
E
Strain energy density
ϵ

- **Resilience** is the property of a material to absorb energy when it is deformed elastically and then, upon unloading to have this energy recovered.
- The total strain energy stored in the body is generally known as resilience.
- The maximum energy which can be stored in a body up-to elastic limit is called **proof resilience**

- Modulus of resilience is the energy that can be absorbed per unit volume without creating a permanent distortion. It can be calculated by integrating the stress-strain curve from zero to the elastic limit and dividing by the original volume of the specimen

Ex:12. A cube of mild steel is subjected to a uniform uniaxial stress as shown;

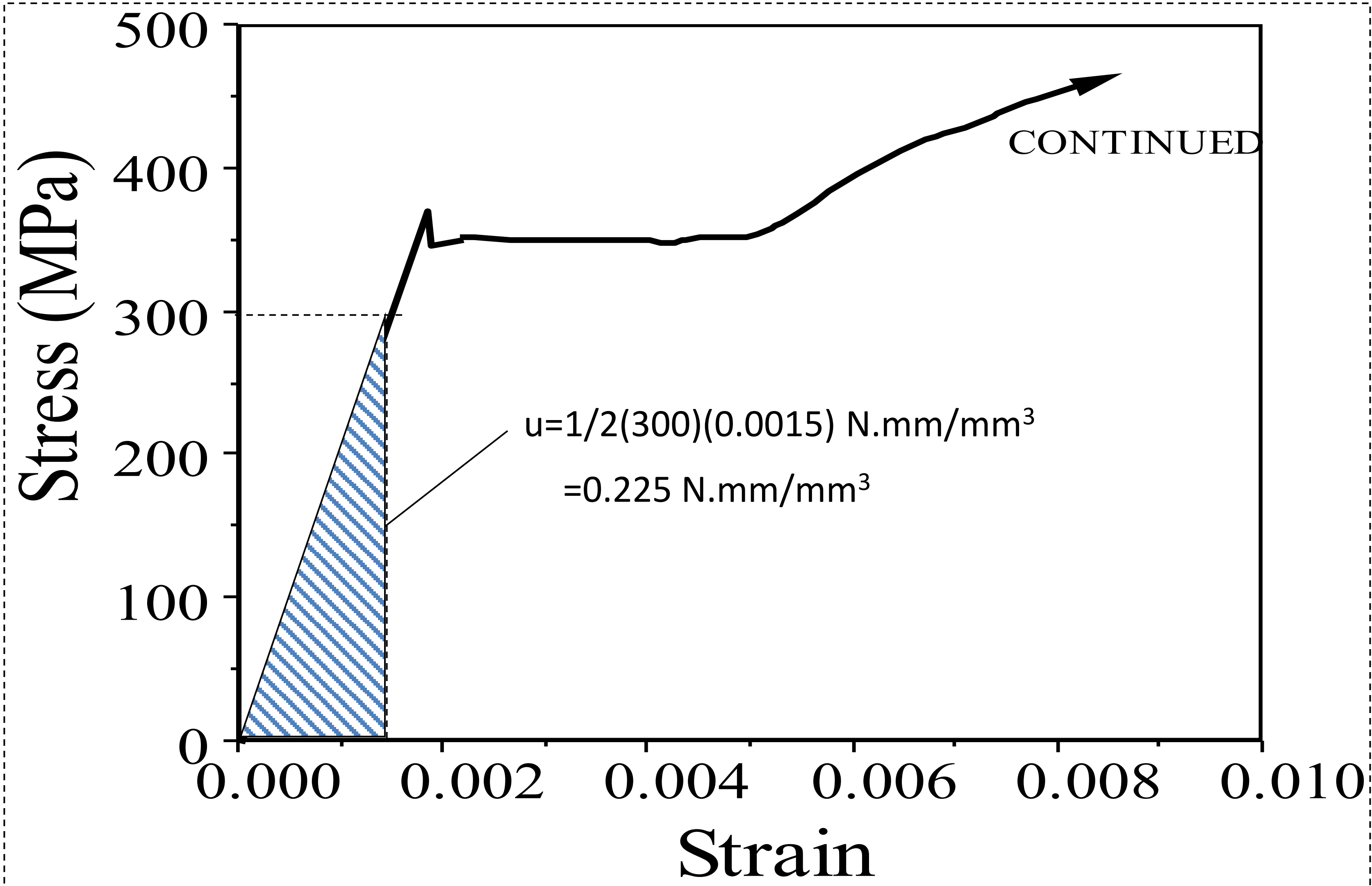
Determine the strain energy density in the cube when:



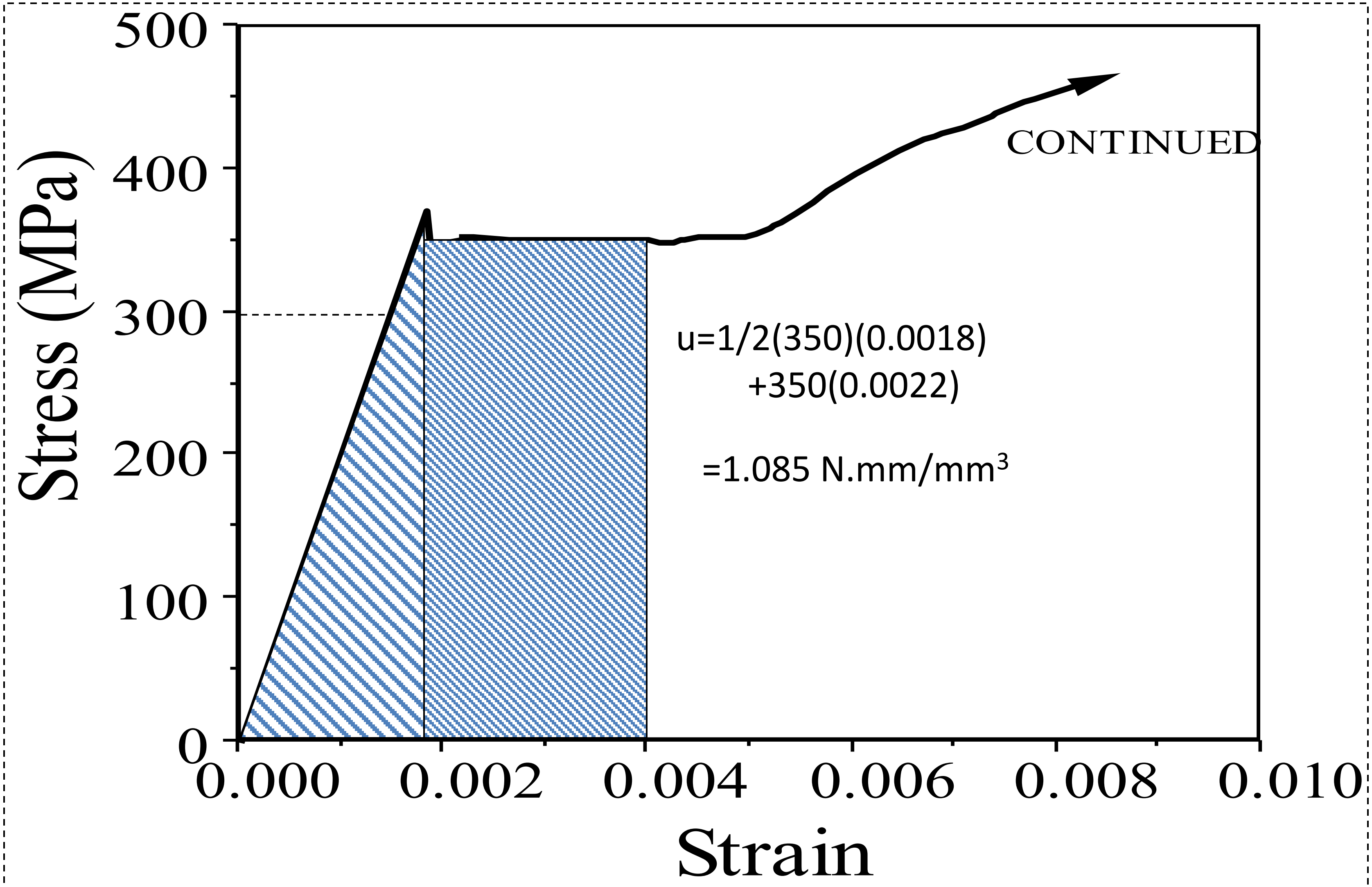
(a)

(a) the stress is 300 MPa; (b) the strain in the x-direction is 0.004

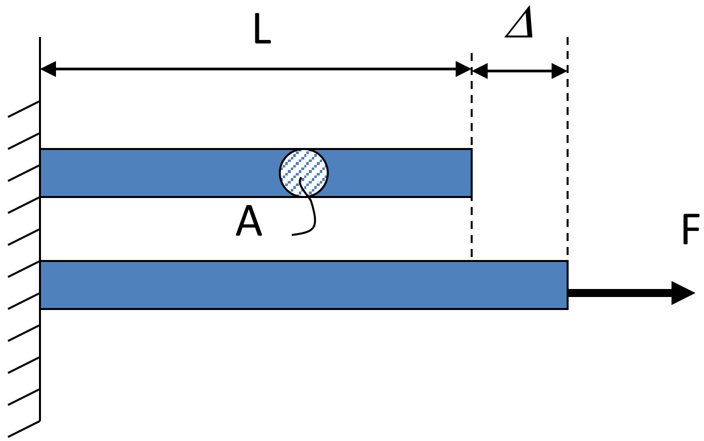
(a) For a linear elastic material



(b) Consider elastic- plastic



Strain Energy for axially loaded bar



$$\sigma_{\text{axial}} = \frac{F}{A}; \Delta = \frac{FL}{AE};$$

$$U = \frac{1}{2} F\Delta = \frac{F^2 L}{2AE}$$

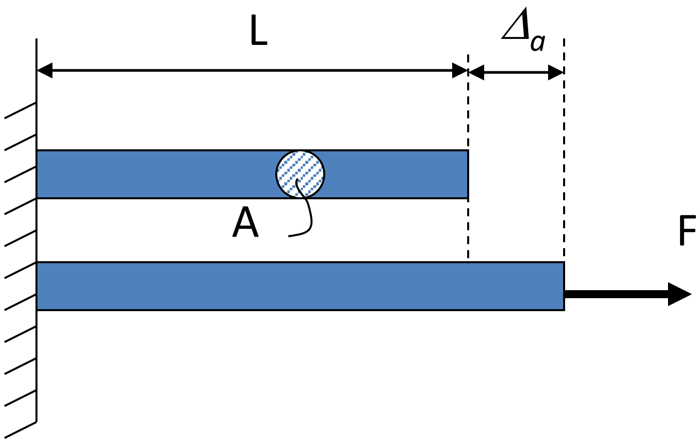
F= Axial Force

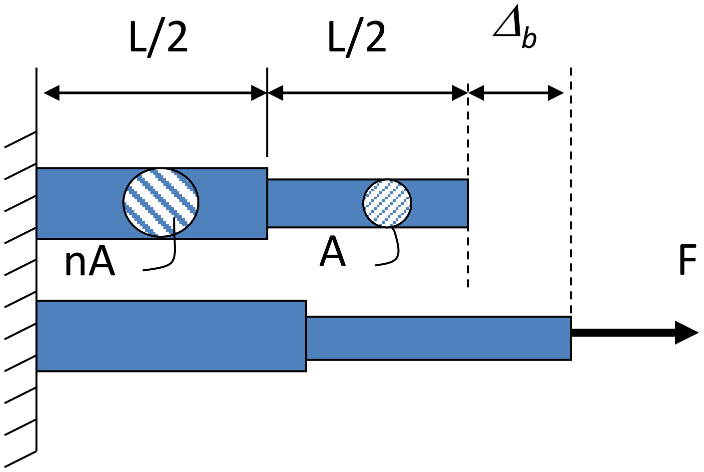
A = Cross-Sectional Area
Perpendicular to *"F"*

E = Modulus of elasticity,

L = Original Length of Bar,

Comparison of Energy Stored in Straight and Stepped bars





(a) $$U = \frac{F^2 L}{2AE}$$

(b) $$U = \frac{F^2 L/2}{2AE} + \frac{F^2 L/2}{2nAE}$$

$$= \frac{F^2 L}{2AE}\left(\frac{1+n}{2n}\right)$$

UNIT–2

BEAMS – LOADS AND STRESSES

Beam - Definition

A structural member which is long when compared with its lateral dimensions, subjected to transverse forces so applied as to induce bending of the member in an axial plane, is called a beam.

In most cases, the loads are perpendicular to the axis of the beam. Such a *transverse loading* causes only bending and shear in the beam.

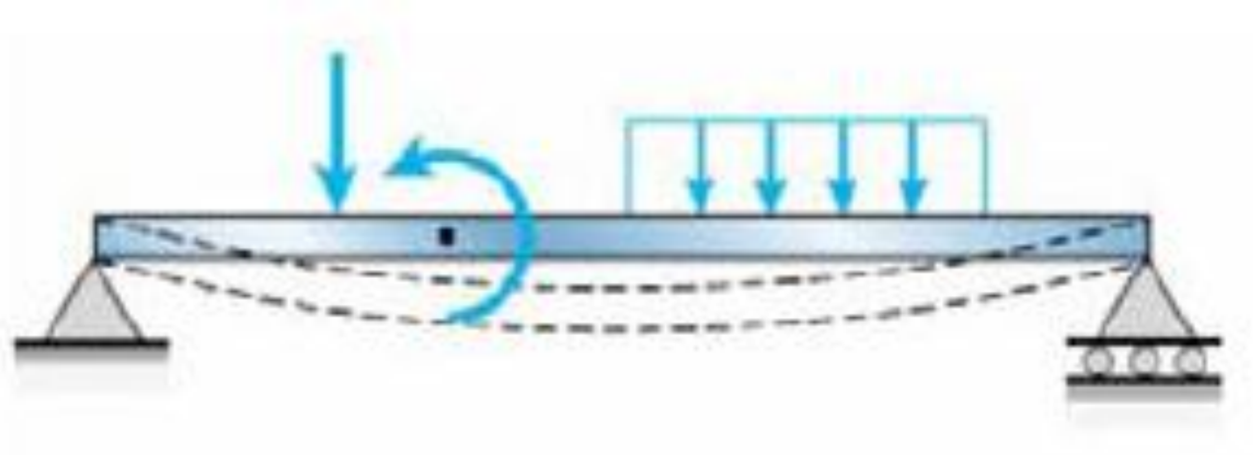
When the loads are not at a right angle to the beam, they also produce axial forces in the beam.

Types of loads

(a) Concentrated loads

(b) Distributed loads

(c) couples



Types of beams

- Beams are usually described by the manner in which they are supported and also based on degree of statical indeterminacy

- Based on support condition, they are classified as

(i) simply supported beam or a simple beam
(ii) cantilever beam
(iii) beam with overhangs
(iv) Continuous beam
(v) Propped cantilever
(vi) Fixed beam

simply supported beam: A beam with a pin support at one end and a roller support at the other is called a simply supported beam or a simple beam

cantilever beam: A beam which is fixed at one end and free at the other end. The free end is free to translate and rotate unlike the fixed end that can do neither.

Overhangs beams: A simply supported beam with overhangs at the end/ends.

Continuous Beam: A beam supported by more than two supports

Propped cantilever : A beam with one end fixed and the other end hinged or supported on rollers.

Fixed beam: A beam which is fixed at both the ends

- Based on the degree of statical indeterminacy beams are classified as

(i) statically determinate beams

Ex: simply supported beams with or with our overhangs and

cantilever beams

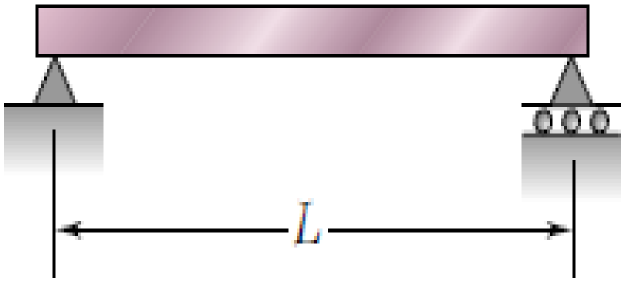
(ii) statically indeterminate beams

Ex: Continuous beams,

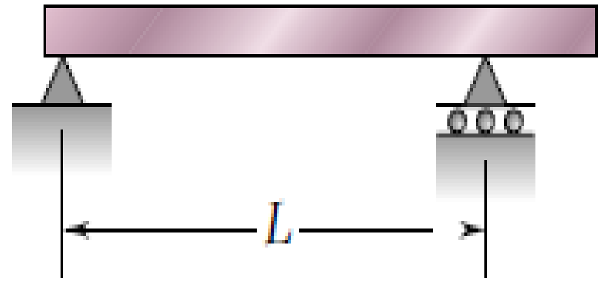
Propped cantilever and

Fixed beams

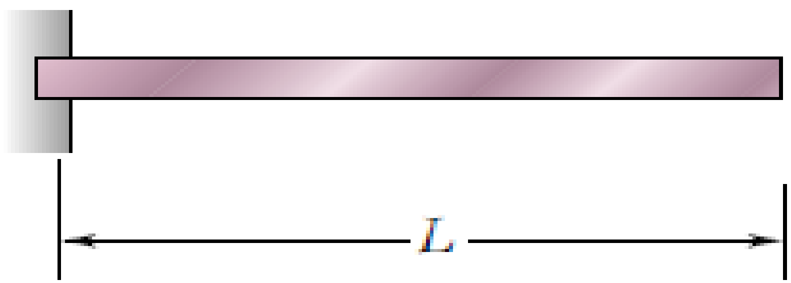
(i) statically determinate beams



(a) Simply supported beam

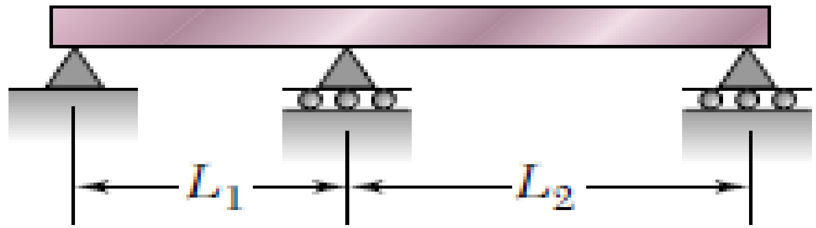


(b) Overhanging beam

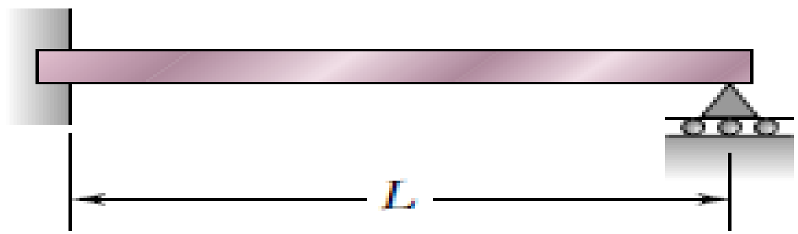


(c) Cantilever beam

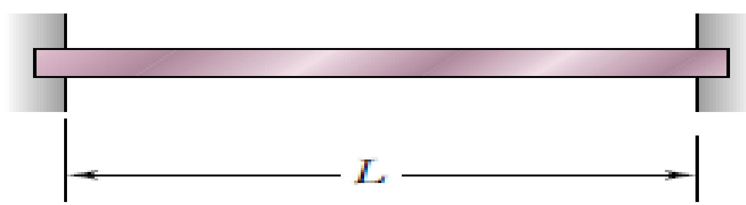
• (ii) Statically Indeterminate Beams



(*d*) Continuous beam



(*e*) Beam fixed at one end
and simply supported
at the other end



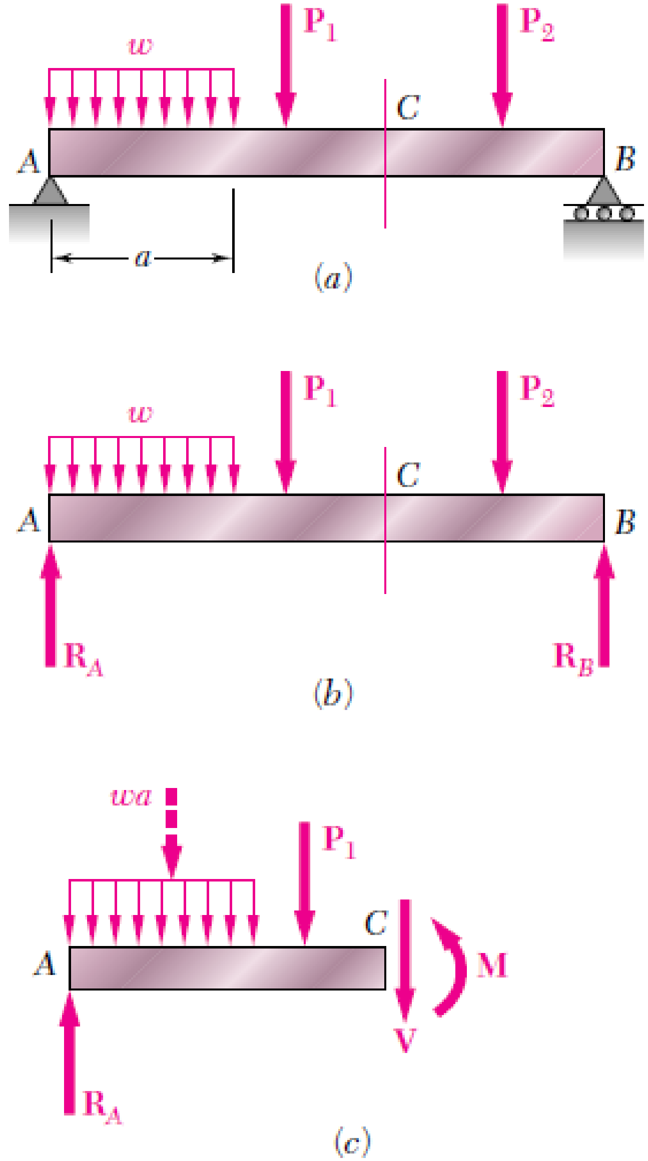
(*f*) Fixed beam

SHEAR FORCE AND BENDING MOMENT

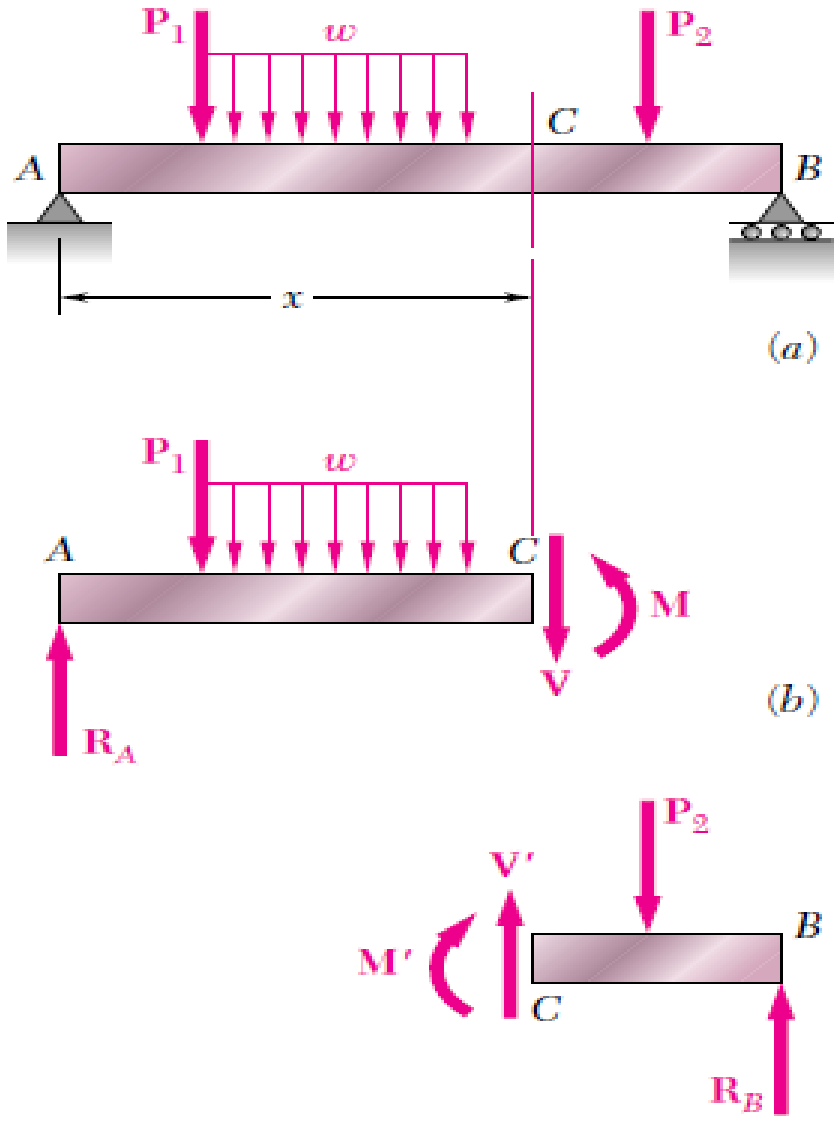
- When a beam is loaded by forces or couples, stresses and strains are created throughout the interior of the beam.
- To determine these stresses and strains, the internal forces and internal couples that act on the cross sections of the beam must be found.
- To find the internal quantities, consider a simply supported beam as in Figure below

- Cut the beam at a cross-section C located at a distance x from end A and consider the free body of left or right part.
- The free body is held in equilibrium by the forces and by the stresses that act over the cut cross section.
- The resultant of the stresses must be such as to maintain the equilibrium of the free body.
- The resultant of the stresses acting on the cross section can be reduced to a shear force V and a bending moment M.

Concept



- Concept



- The stress resultants in statically determinate beams can be calculated from equations of equilibrium

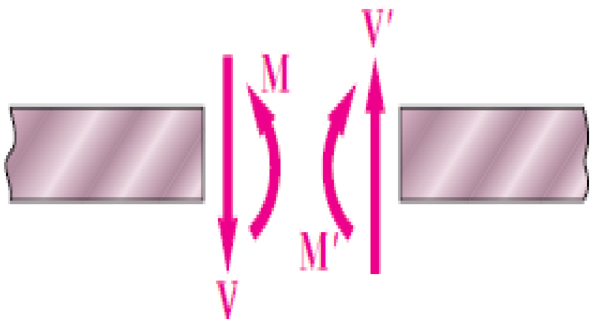
Definitions:

- **Shear Force:** is the algebraic sum of the vertical forces acting to the left or right of the cut section

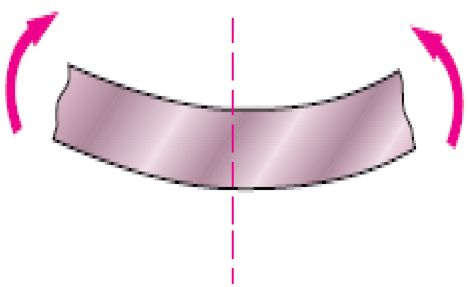
- **Bending Moment:** is the algebraic sum of the moment of the forces to the left or to the right of the section taken about the section

- Sign convention

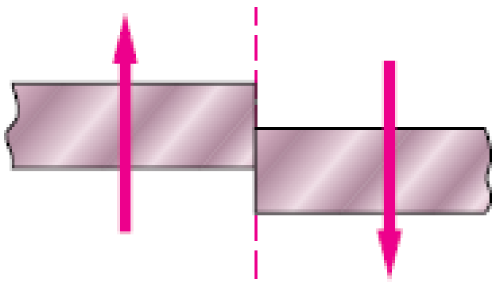
  The shear V and the bending moment M at a given point of a beam are said to be positive when the internal forces and couples acting on each portion of the beam are directed as shown in Fig. a.



(a) Internal forces
(positive shear and positive bending moment)



(c) Effect of external forces
(positive bending moment)



(b) Effect of external forces
(positive shear)

- These conventions can be more easily remembered if we note that

- **1.** *The shear at any given point of a beam is positive when the* **external** *forces (loads and reactions) acting on the beam tend to shear off the beam at that point as indicated in Fig.b*.
- **2.** *The bending moment at any given point of a beam is positive when the* **external** *forces acting on the beam tend to bend the beam at that point as indicated in Fig. c.*

- **Ex: 13**
- Draw the shear and bending-moment diagrams for a simply supported beam *AB* of span *L* subjected to a single concentrated load **P** at it midpoint *C*

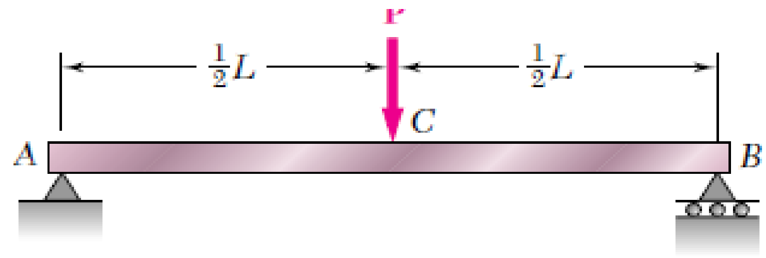
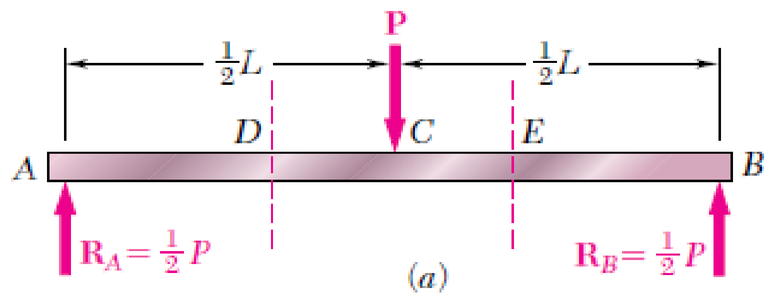
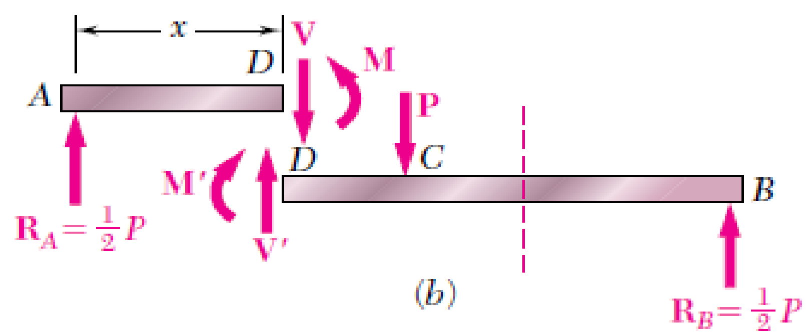
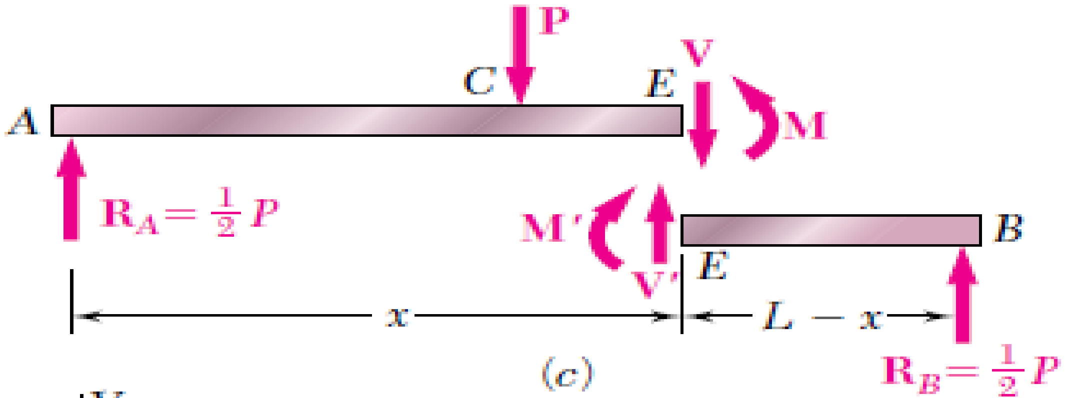


Fig. 5.8

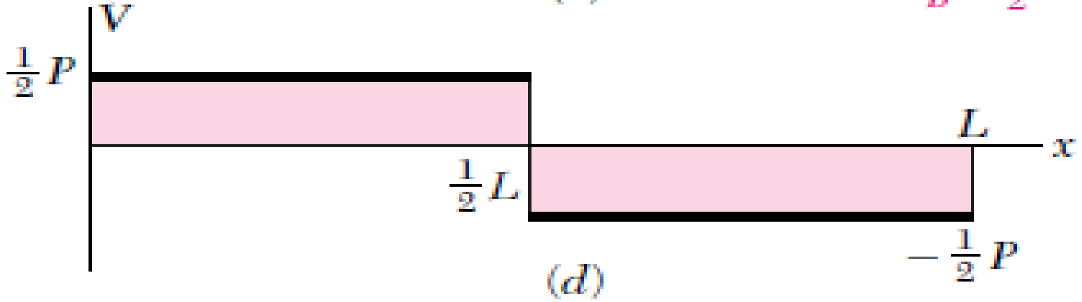




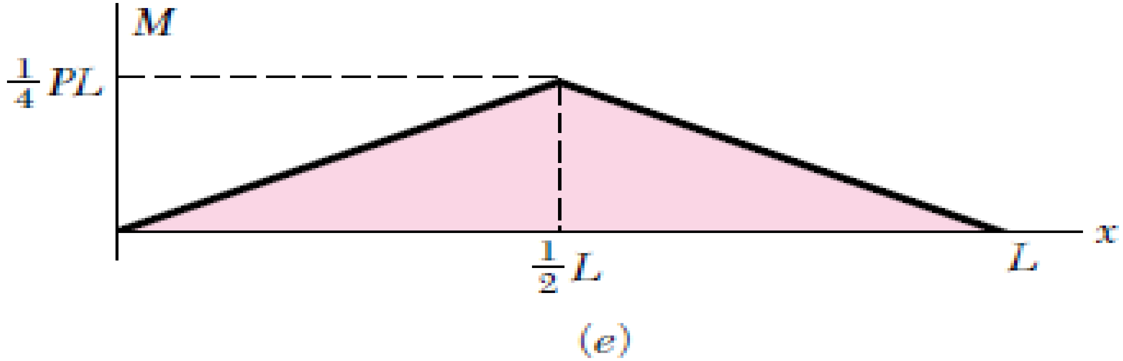
Beam


P
C
E
V
M
A
$R_A = \frac{1}{2}P$
M'
V'
E
B
x
L − x
(c)
$R_B = \frac{1}{2}P$

SFD


V
$\frac{1}{2}P$
L
x
$\frac{1}{2}L$
$-\frac{1}{2}P$
(d)

BMD


M
$\frac{1}{4}PL$
x
$\frac{1}{2}L$
L
(e)

- **Ex: 14**
- Draw the shear and bending-moment diagrams for the simply supported beam shown in Fig. 5.13 and determine the maximum value of the bending moment.

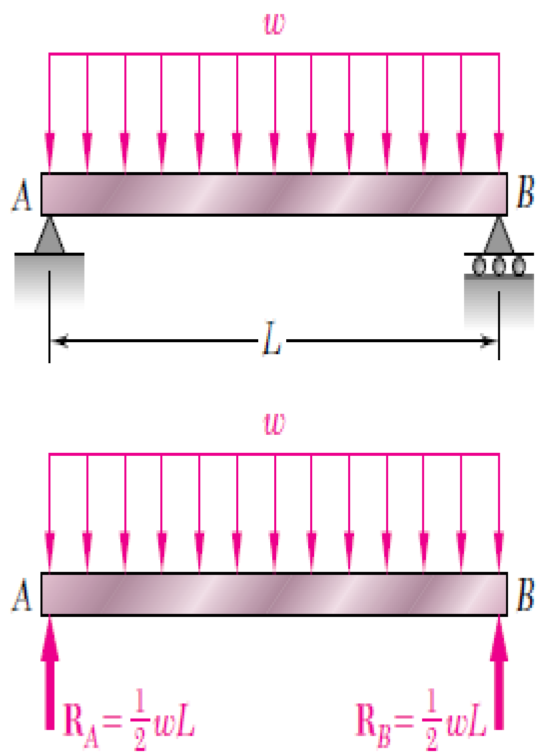
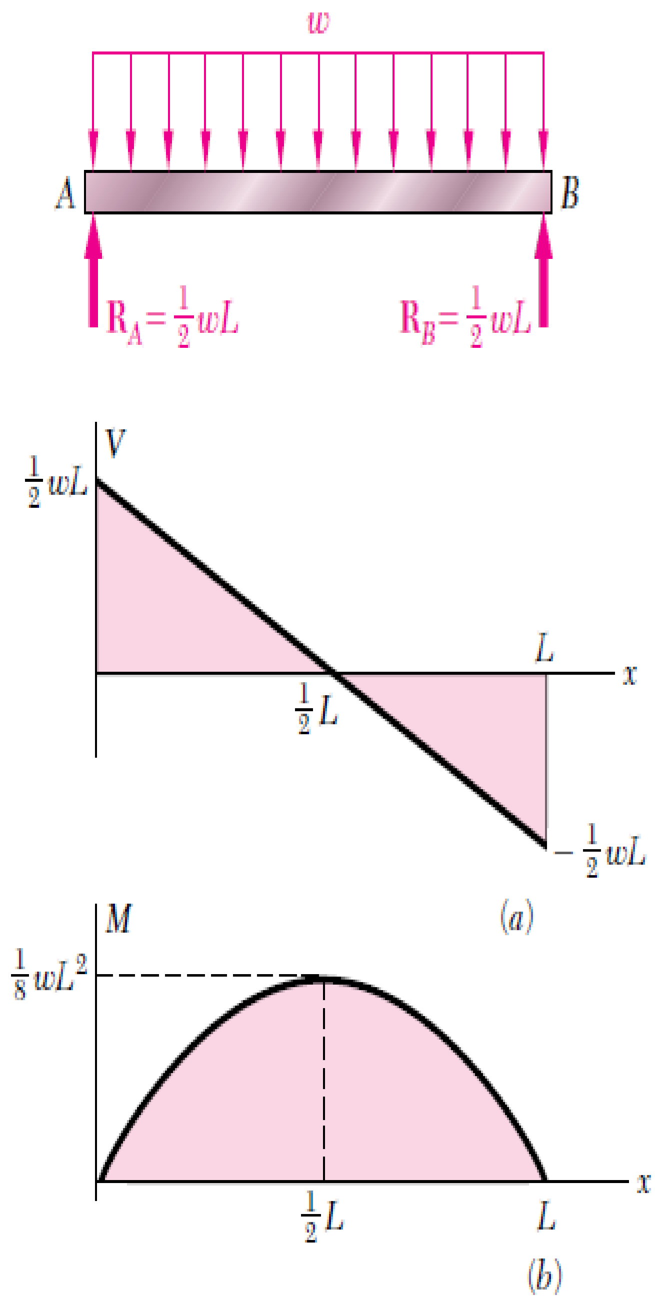


Fig. 5.13

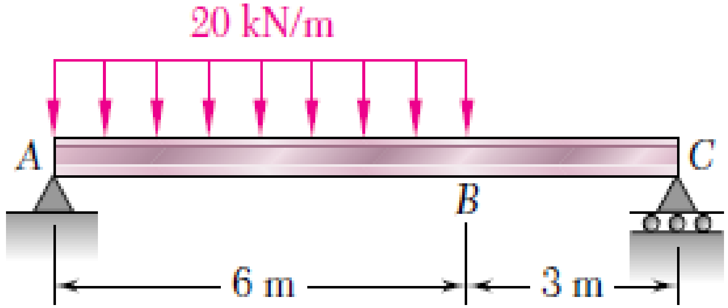
Beam

SFD

BMD



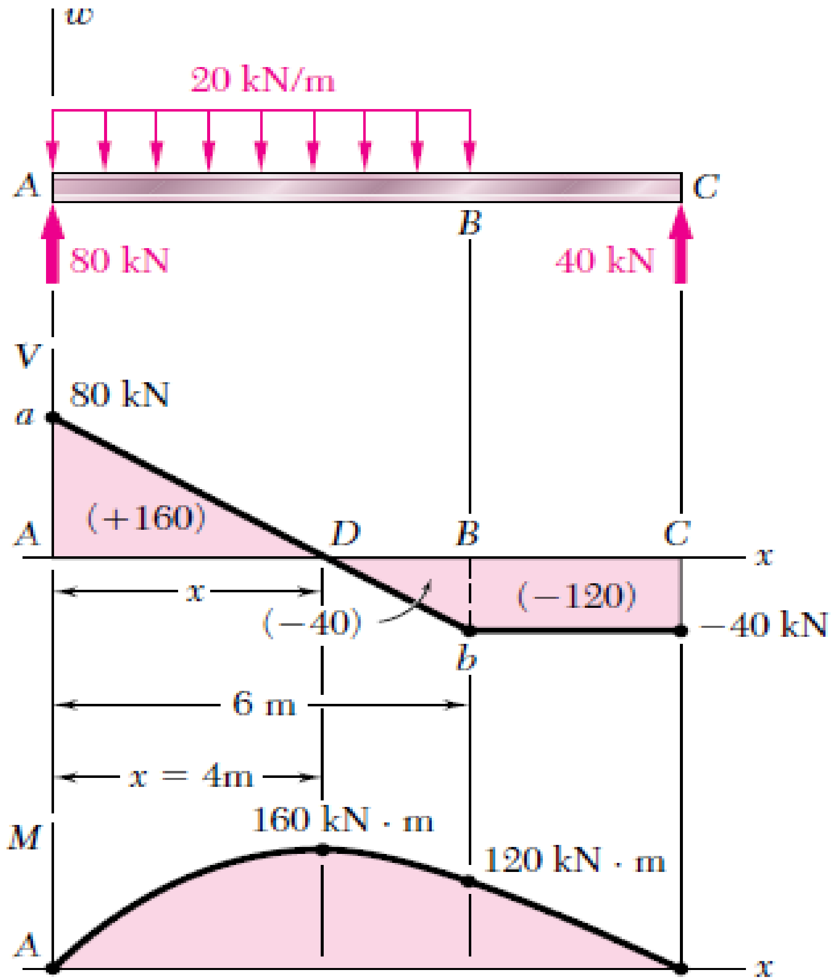
- Ex.15



Beam

SFD

BMD



- **RELATIONS AMONG LOAD, SHEAR, AND BENDING MOMENT**

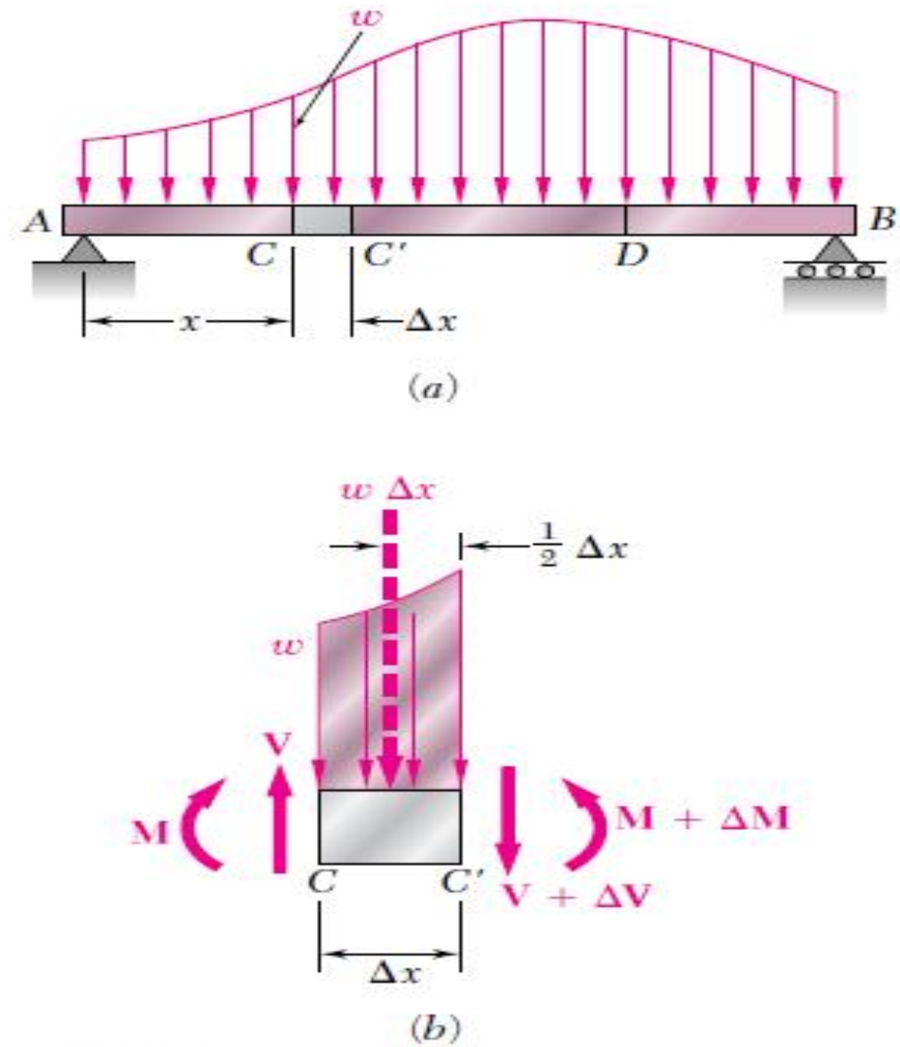


Fig. P5.12

Relations between Load and Shear. Writing that the sum of the vertical components of the forces acting on the free body CC' is zero, we have

$$+\uparrow \Sigma F_y = 0: \qquad V - (V + \Delta V) - w\,\Delta x = 0$$

$$\Delta V = -w\,\Delta x$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\boxed{\frac{dV}{dx} = -w} \qquad (5.5)$$

Relations between Shear and Bending Moment. Returning to the free-body diagram of Fig. 5.12*b*, and writing now that the sum of the moments about C' is zero, we have

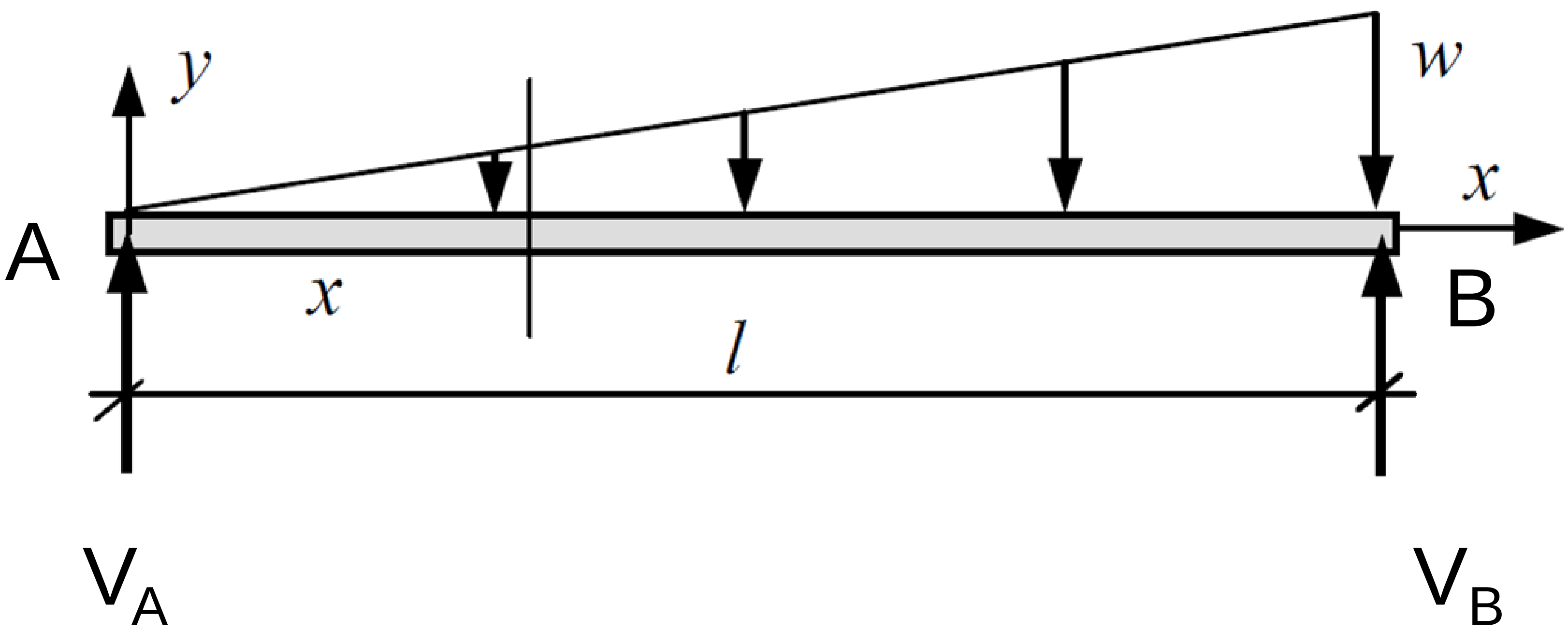
$$+\circlearrowleft \Sigma M_{C'} = 0: \qquad (M + \Delta M) - M - V\,\Delta x + w\,\Delta x \frac{\Delta x}{2} = 0$$

$$\Delta M = V\,\Delta x - \frac{1}{2} w\,(\Delta x)^2$$

Dividing both members of the equation by Δx and then letting Δx approach zero, we obtain

$$\boxed{\frac{dM}{dx} = V} \qquad (5.7)$$

Ex.16 Draw the SFD and BMD for the simply Supported beam subjected to a triangular loading as shown below.



Step 1 Determination of support reactions

To find $V_{B,}$ apply moment equilibrium condition at A, i.e., $\sum M_A = 0$ +

$$\frac{w\,l}{2} \times \frac{2}{3} l - V_B l = 0$$

$$V_B = \frac{wl}{3}$$

To find V_A, $\sum V = 0$ +

$$V_A + V_B - \frac{w\,l}{2} = 0$$

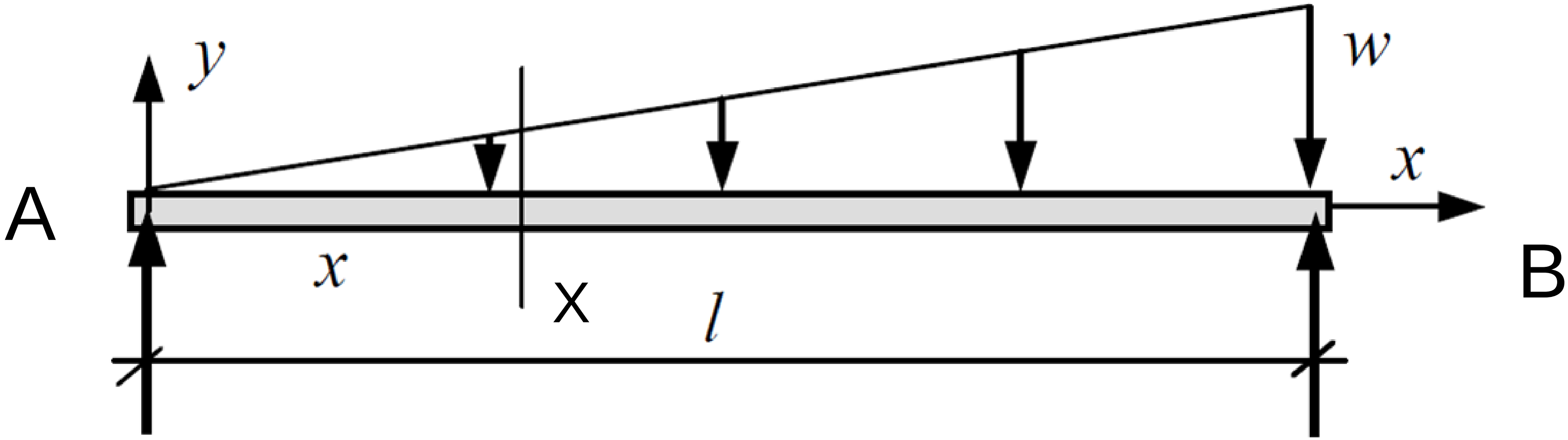
$$V_A = \frac{w\,l}{2} - V_B$$

$$= \frac{w\,l}{2} - \frac{wl}{3}$$

$$= \frac{wl}{6}$$

V_A = wl/6

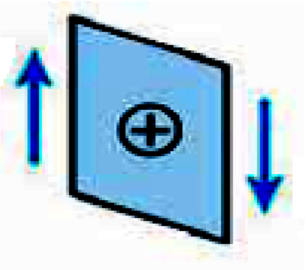
V_B = wl/3



$$V_A = \frac{wl}{6}$$

$$V_B = \frac{wl}{3}$$

Step 2 Shear force diagram



$$(F_A)_L = 0$$

$$(F_A)_R = \frac{wl}{6}$$

$$(F_B)_L = \frac{-wl}{3}$$

$$(F_B)_R = 0$$

General equation for shear force at a section X, at a distance x from A,

$$F_X = \frac{wl}{6} - \frac{1}{2}\, x\, \frac{w}{l}\, x$$

$$= \frac{wl}{6} - \frac{1}{2}\,\frac{w}{l}\, x^2$$

$$= \frac{wl}{6} - \frac{wx^2}{2l}$$

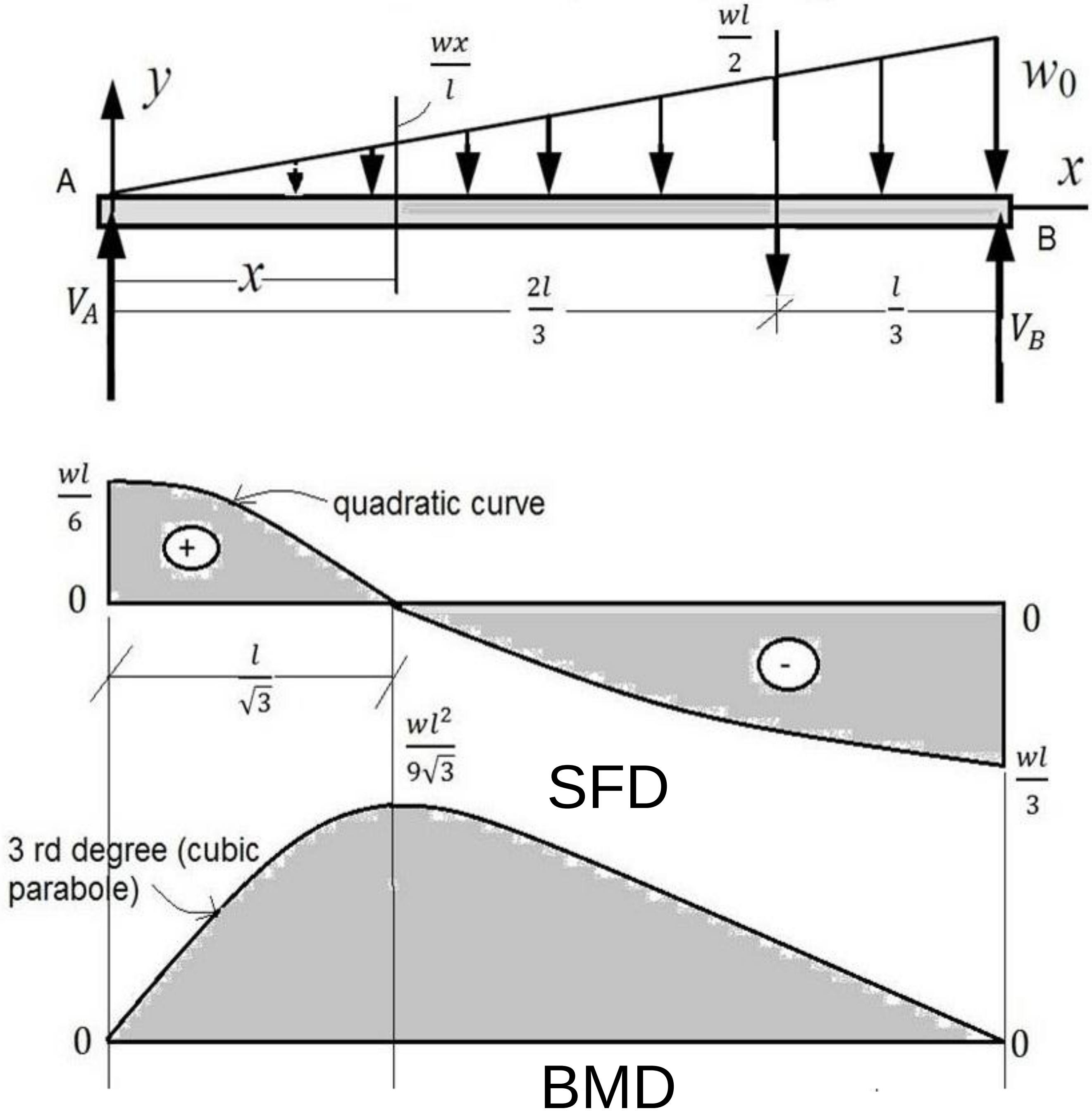
Step 3 Bending Moment Diagram

$$M_A = 0$$

$$M_B = 0$$

$$M_X = \frac{wl}{6} \times x - \left(\frac{1}{2}\, x\, \frac{w}{l}\, x\right) \frac{x}{3}$$

$$= \frac{wlx}{6} - \frac{wx^3}{6l}$$



wl/2
wx/l
y
w0
x
A
B
x
VA
2l/3
l/3
VB
wl/6
quadratic curve
+
0
0
-
l/√3
wl²/(9√3)
wl/3
SFD
3 rd degree (cubic parabole)
0
0
BMD

Determination of Maximum bending moment

For M_x to be max,

$$\frac{dM_x}{dx} = 0$$

$$\frac{wl}{6} - \frac{wx^2}{2l} = 0$$

$$x^2 = \frac{wl}{6} \times \frac{2\,l}{w} = \frac{l^2}{3}$$

$$x = \pm\frac{l}{\sqrt{3}}$$

Since $-\dfrac{l}{\sqrt{3}}$ is not admissible,

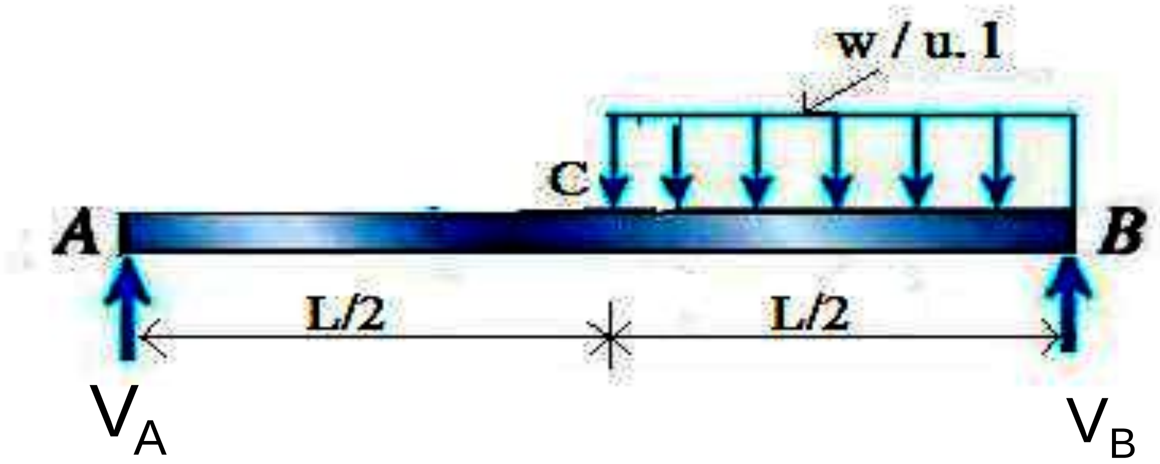
Therefore, $x = \dfrac{l}{\sqrt{3}}$

$$M_{max} = \frac{wl}{6}\left(\frac{l}{\sqrt{3}}\right) - \frac{w}{6l}\left(\frac{l}{\sqrt{3}}\right)^3$$

$$= \frac{wl^2}{6\sqrt{3}} - \frac{wl^2}{18\sqrt{3}}$$

$$= \frac{wl^2}{9\sqrt{3}}$$

Ex.17 Draw the SFD and BMD for the simply Supported beam shown below.



Step 1: Determination of support reactions

To find $V_{B,}$ apply moment equilibrium condition

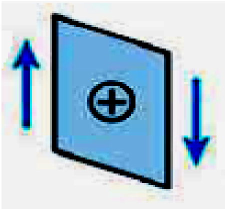
at A, i.e., $\sum M_A = 0$

$$w \times \frac{l}{2} \times \frac{3l}{4} - V_B \times l = 0$$

$$V_B = \frac{3wl}{8}$$

$$\therefore \; V_A = \frac{wl}{2} - \frac{3wl}{8} = \frac{wl}{8}$$

Step 2 Shear force diagram (SFD)



$$(F_A)_L = 0$$

$$(F_A)_R = \frac{wl}{8}$$

$$F_C = \frac{wl}{8}$$

$$(F_B)_L = \frac{-3wl}{8}$$

$$(F_B)_R = 0$$

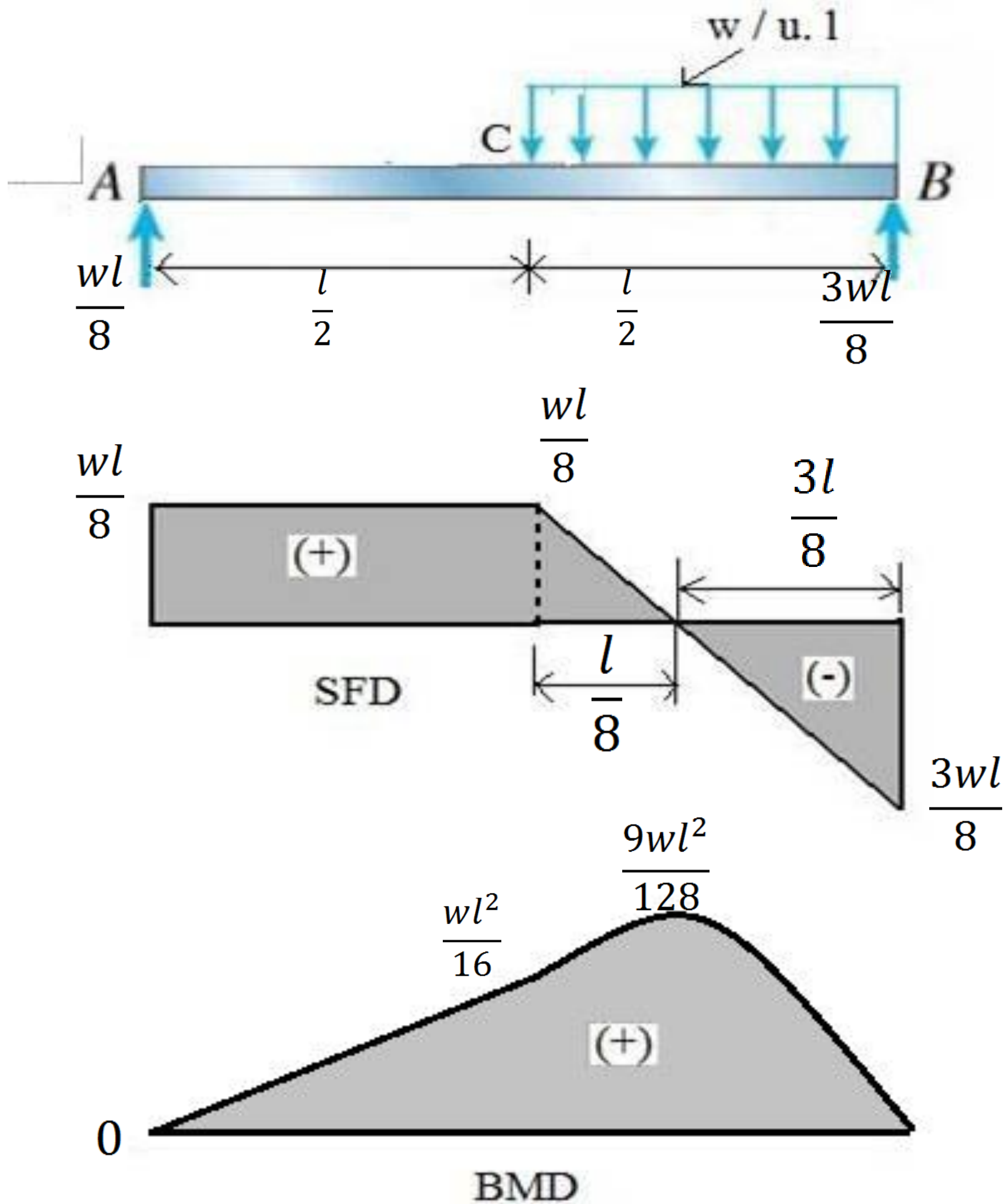
Step 3 Bending Moment Diagram (BMD)

$$M_A = 0 \qquad M_B = 0$$

$$M_C = \frac{wl}{8} \times \frac{l}{2} \quad = \frac{wl^2}{16}$$

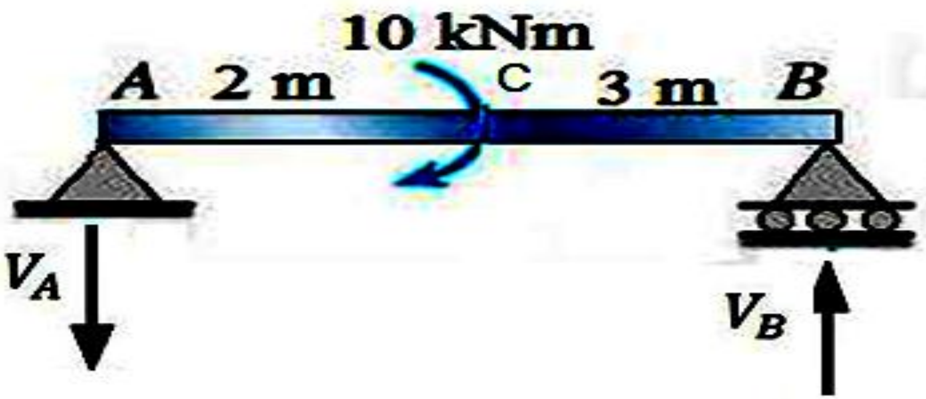
$$M_{max} = \frac{3wl}{8} \times \frac{3l}{8} - \frac{3wl}{8} \times \frac{3l}{16}$$

$$= \frac{9wl^2}{128}$$



w / u. l
C
A
B
$\frac{wl}{8}$
$\frac{l}{2}$
$\frac{l}{2}$
$\frac{3wl}{8}$
$\frac{wl}{8}$
$\frac{wl}{8}$
$\frac{3l}{8}$
(+)
$\frac{l}{8}$
(-)
SFD
$\frac{3wl}{8}$
$\frac{9wl^2}{128}$
$\frac{wl^2}{16}$
(+)
0
BMD

Ex : 18 Draw the SFD and BMD for the Supported beam shown below



<u>Soln:</u>

Step 1: Determination of support reactions

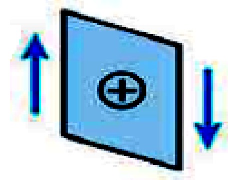
To find the reaction at B, apply moment equilibrium condition at A, $\sum M_A = 0$ +

i.e., $10 - V_B \times 5 = 0$

$V_B = 10/5 = 2$ kN ↑ , and $V_A = 10/5 = 2$ kN ↓

Step 2 Shear force diagram (SFD)

$(F_A)_L = 0,$ $(F_A)_R = -2$ kN



$F_C = -2$ kN

$(F_B)_L = -W,$ $(F_B)_R = 0$

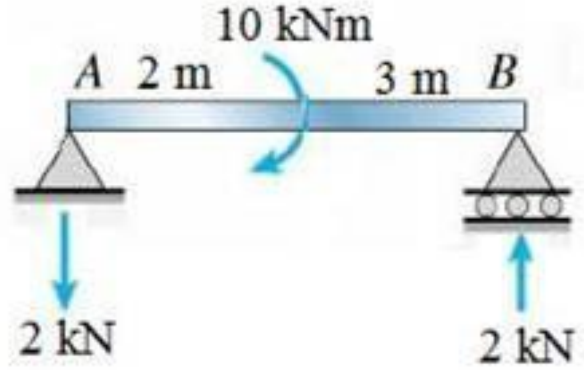
Step 3 Bending Moment Diagram (BMD)

$M_A = 0$

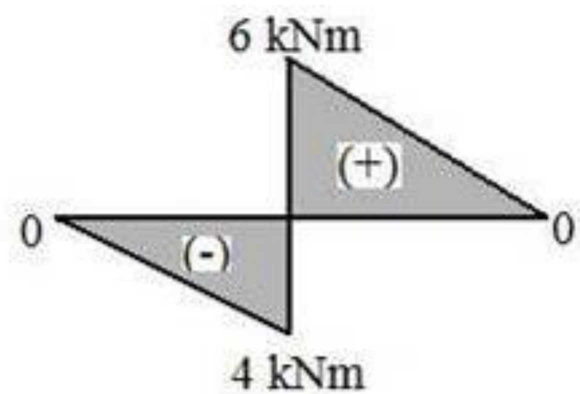
$(M_C)_L = -2 \times 2 = -4$ kNm

$(M_C)_R = 2 \times 3 = 6$ kNm

$M_B = 0$

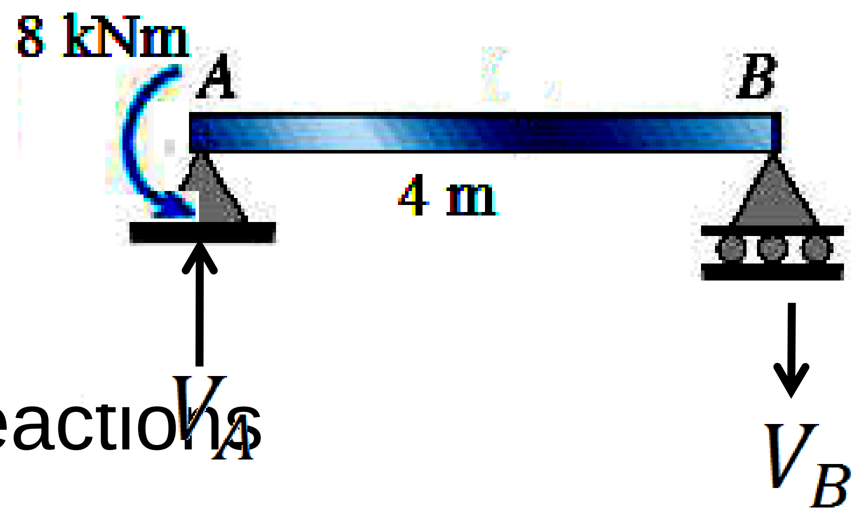


SFD



BMD

Ex:19 Draw the SFD and BMD for the Supported beam shown below



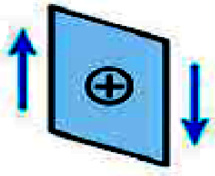
Soln:

Step 1: support reactions

$V_A = 8/4 = 2$ kN

$V_B = 8/4 = 2$ kN

Step 2 Shear force diagram (SFD)



$(F_A)_L = 0,$ $(F_A)_R = 2$ kN

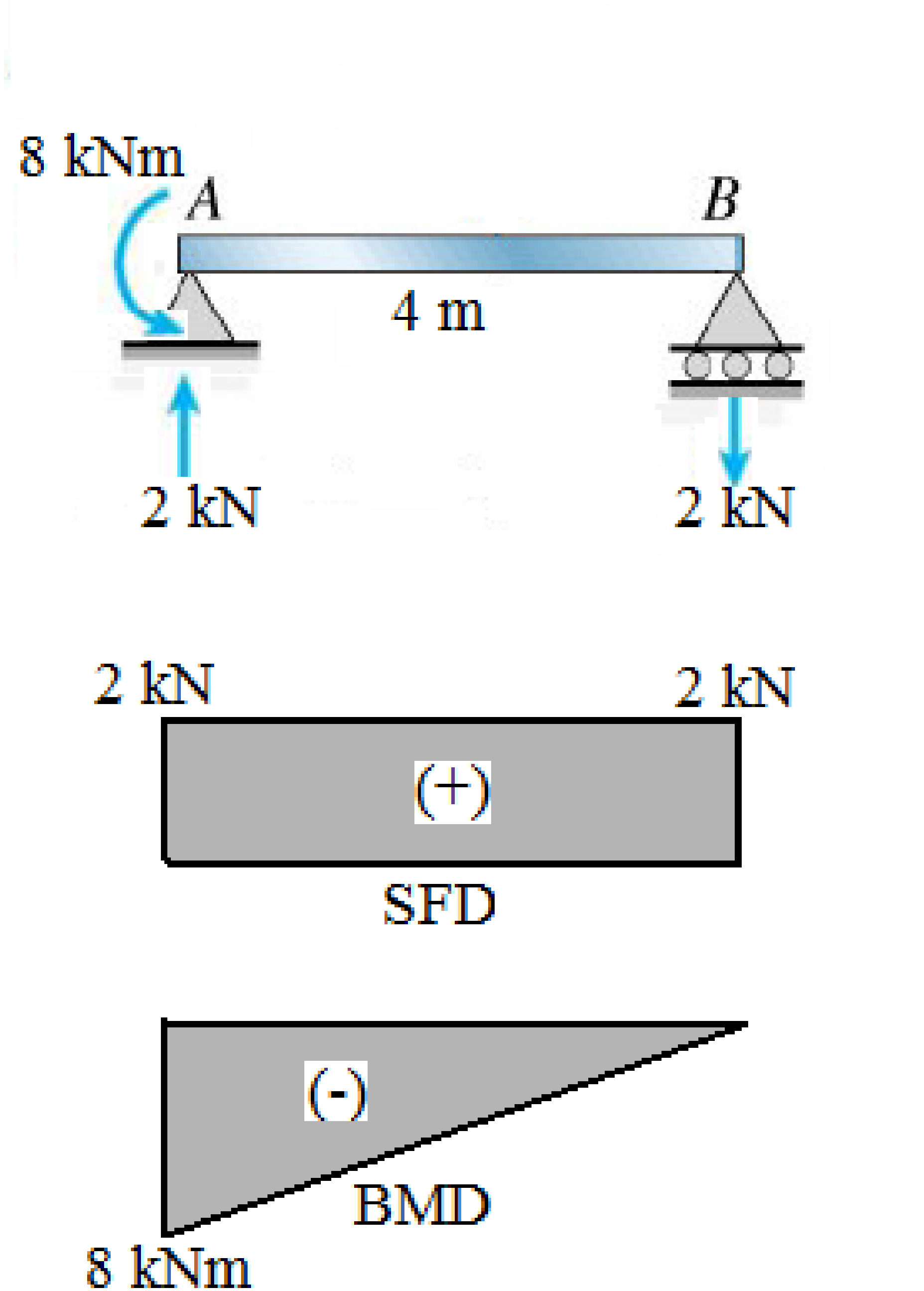
$(F_B)_L = 2$ kN , $(F_B)_R = 0$

Step 3 Bending Moment Diagram (BMD)

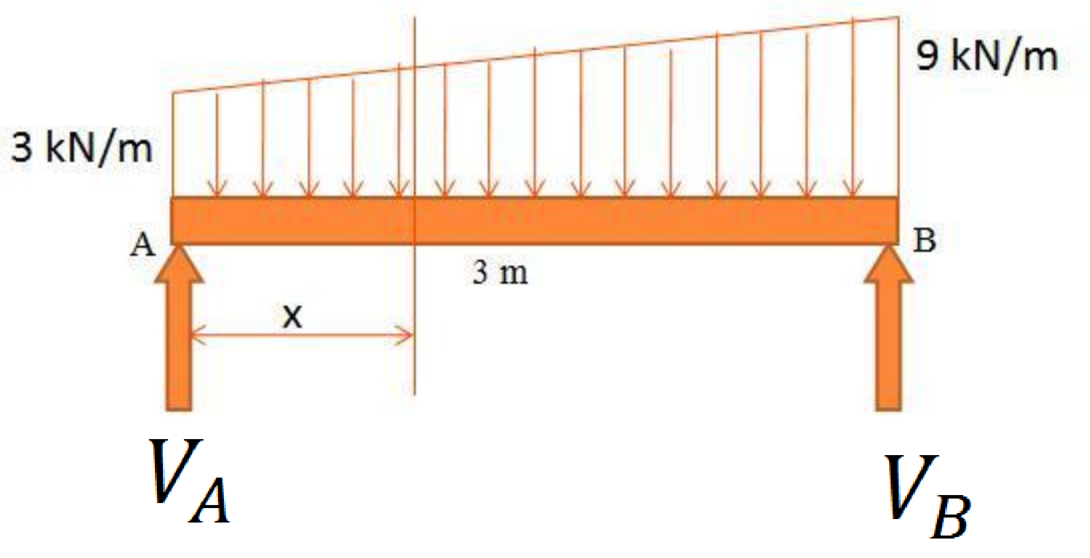
$(M_A)_L = 0$

$(M_A)_R = -8$ kNm

$M_B = 0$


8 kNm
A
B
4 m
2 kN
2 kN
2 kN
2 kN
(+)
SFD
(-)
BMD
8 kNm

Ex:20. Draw the SFD and BMD for the Supported beam shown below subjected to a trapezoidal loading.



<u>Soln:</u>

Step 1: Determination of support reactions

To find V_B, apply $\sum M_A = 0$

$$\frac{1}{2} \times 3 \times 12 \times \frac{3}{3} \frac{(3 + 2 \times 9)}{(3 + 9)} - 3V_B = 0$$

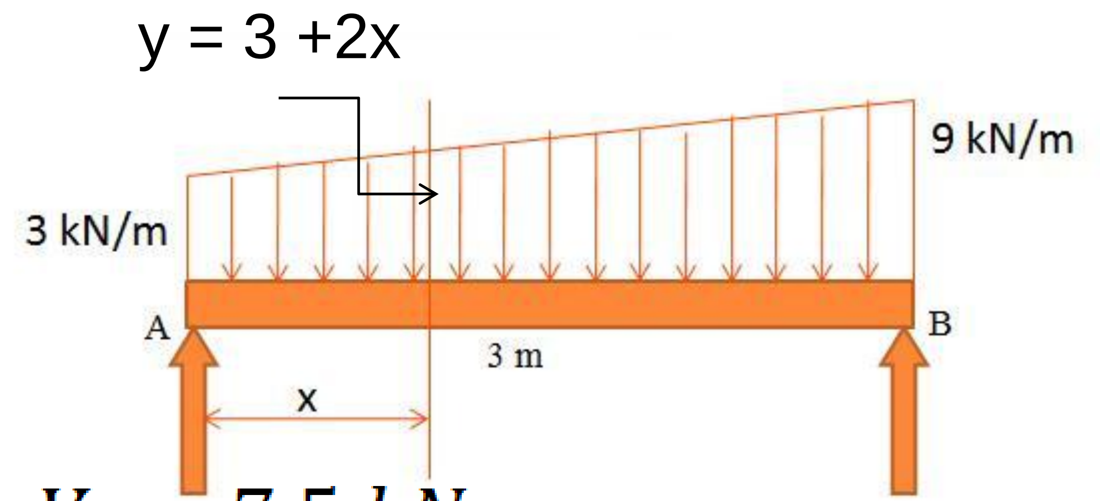
$$V_B = 10.5\ kN$$

$$(or)$$

$$\left(3 \times 3 \times \frac{3}{2}\right) + \left(\frac{1}{2} \times 3 \times 6 \times 2\right) - 3V_B = 0$$

$$V_B = 10.5\ kN$$

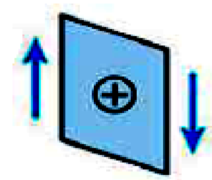
$$\therefore V_A = 7.5\ kN$$



$V_A = 7.5\ kN$

$V_B = 10.5\ kN$

Step 2 Shear force diagram (SFD)



$$(F_A)_L = 0$$

$$(F_A)_R = 7.5\ kN$$

$$(F_B)_L = -10.5\ kN$$

$$(F_B)_R = 0$$

$$F_X = 7.5 - \frac{1}{2}x\,(6 + 2x)$$

$$= 7.5 - 3x - x^2$$

Position of Zero Shear

$$7.5 - 3x - x^2 = 0$$

$$x = 1.6225\, m\, (or)$$

$$- 4.6225 m$$

(- ve value is not admissible)

Therefore, x = 1.6225m

Step 3 Bending Moment Diagram (BMD)

$$M_A = 0$$

$$M_B = 0$$

$$M_X = 7.5\,x - \frac{3x^2}{2} - \left(\frac{1}{2} \times x \times 2x\right)\frac{x}{3}$$

$$= 7.5x - \frac{3x^2}{2} - \frac{x^3}{3}$$

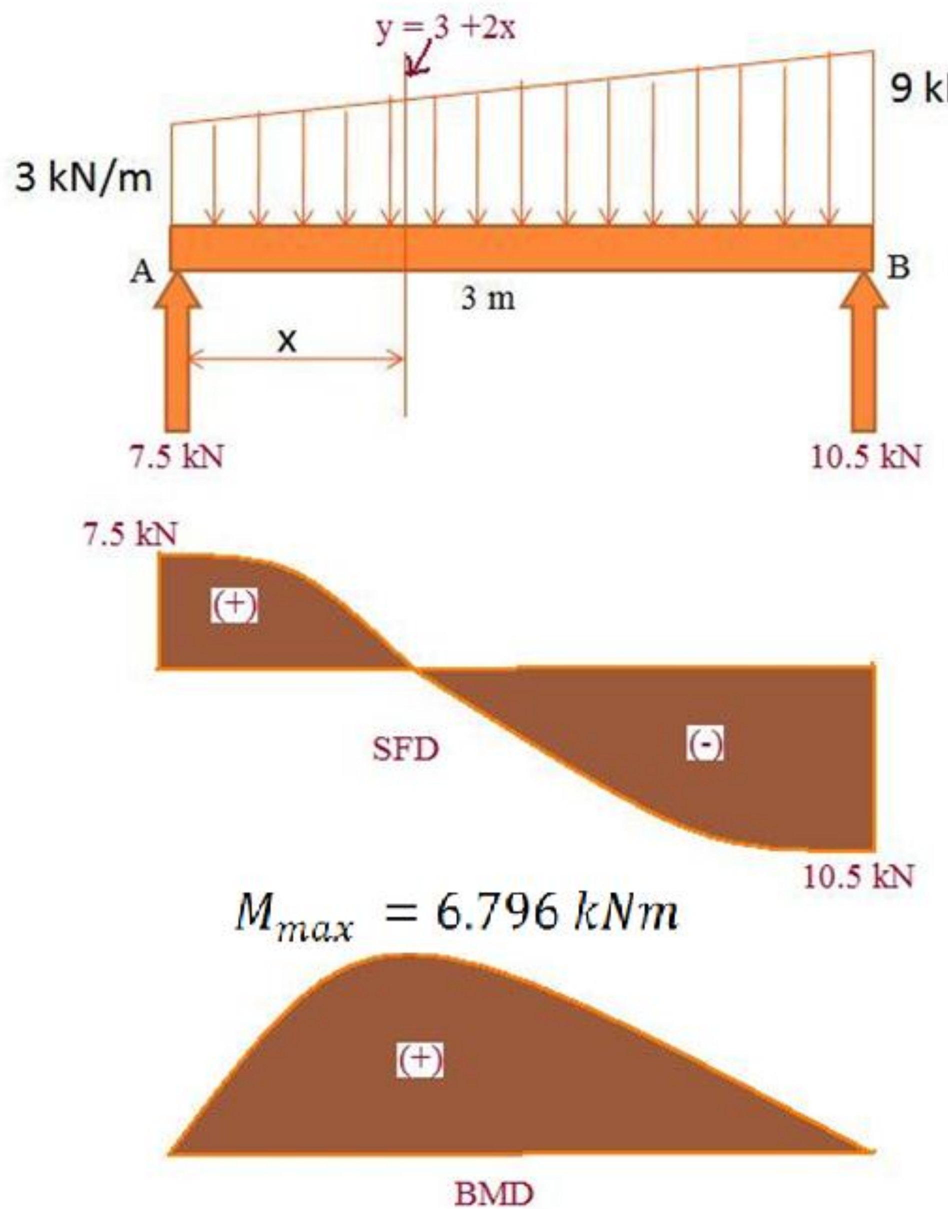
For M_x to be maximum,

$$\frac{dM_x}{dx} = 0$$

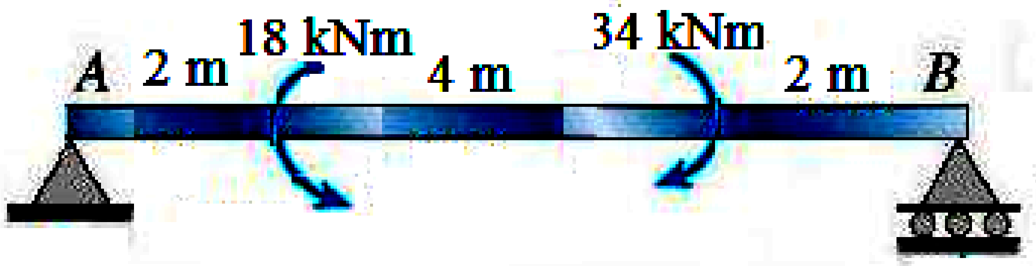
$$7.5 - 3x - x^2 = 0$$

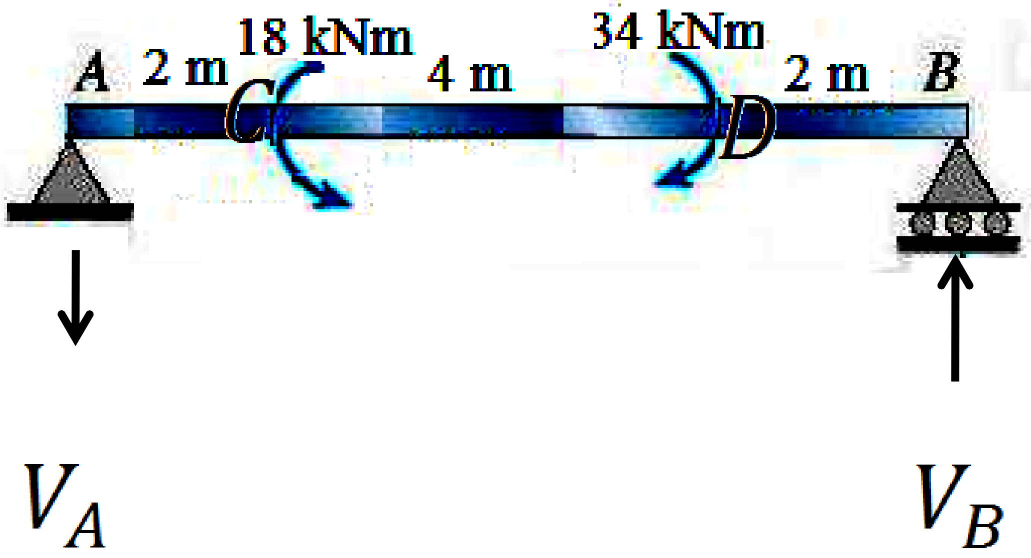
$$x = 1.6225\, m$$

$$M_{max} = 6.796\, kNm$$


y = 3 +2x
9 kN
3 kN/m
A
B
3 m
x
7.5 kN
10.5 kN
7.5 kN
(+)
SFD
(-)
10.5 kN
$M_{max} = 6.796\ kNm$
(+)
BMD

Ex:21. A simply supported beam of span 8m is subjected to two couples as shown in figure below. Draw the shearing force and bending moment diagrams.




18 kNm
34 kNm
A
2 m
C
4 m
D
2 m
B
V_A
V_B

Soln:

Step 1: Determination of support reactions

To find V_B, apply $\sum M_A = 0$

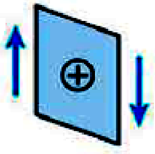
$$-18 + 34 - V_B \times 8 = 0$$

$$V_B = 2 \text{ kN}$$

$$V_A = -2 \text{ kN}$$

$\therefore$

- <u>Step 2</u> Shear force diagram (SFD)



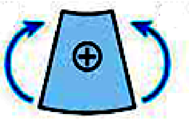
$(F_A)_L = 0,\qquad (F_A)_R = -2\ kN$

$F_C = 0$

$F_D = 0$

$(F_B)_L = -2\ kN\ ,\qquad (F_B)_R = 0$

Step3: Bending Moment Diagram (BMD)



$M_A = 0$

$(M_C)_L = -2 \times 2 = - 4$ kNm

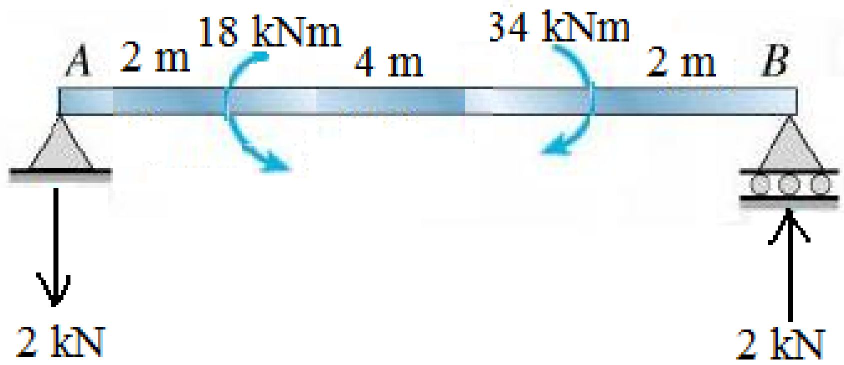
$(M_C)_R = - 2 \times 2 – 18 = - 22$ kNm

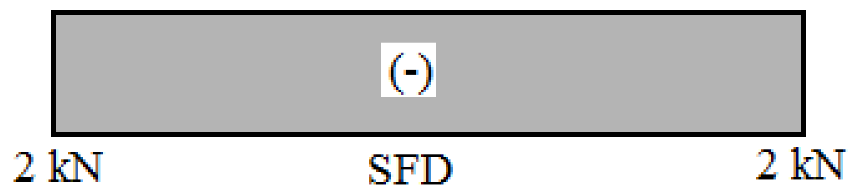
$(M_D)_L = - 2 \times 6 – 18 = - 30$ kNm

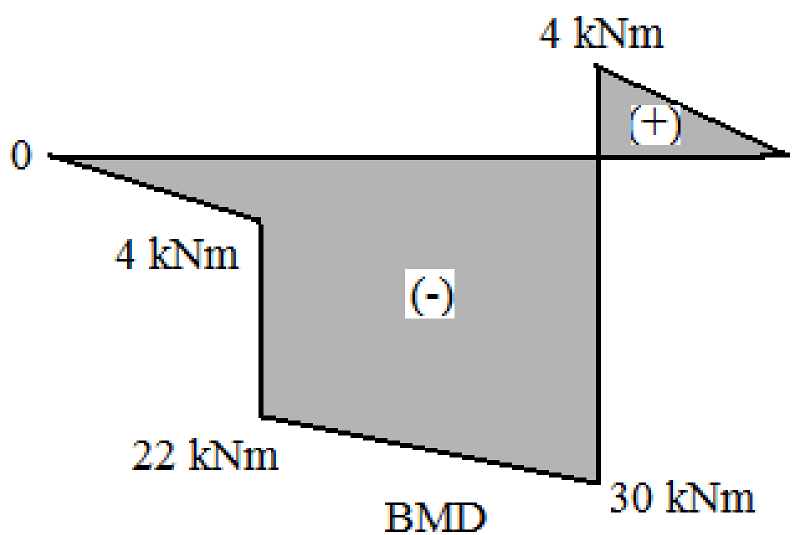
$(M_D)_R = 2 \times 2 = 4$ kNm (or)

$= - 30 + 34 = 4$ kNm

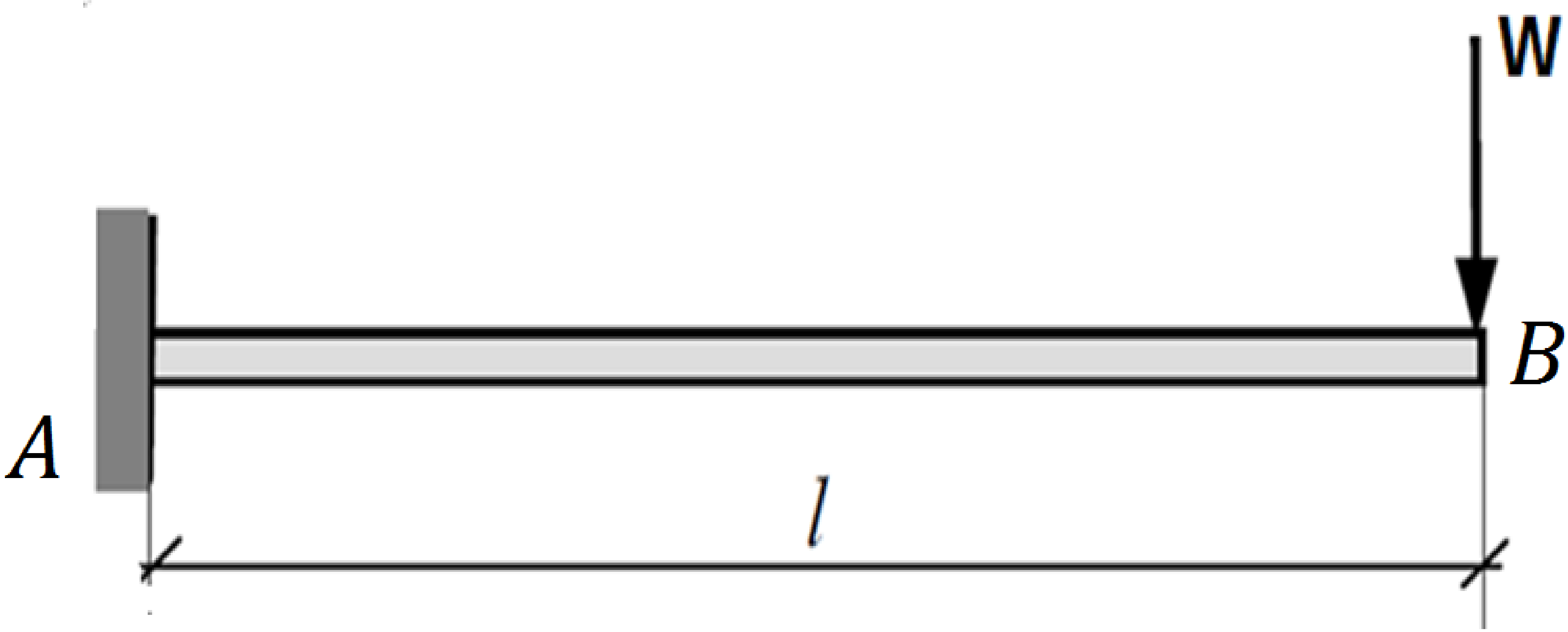
$M_B = 0$

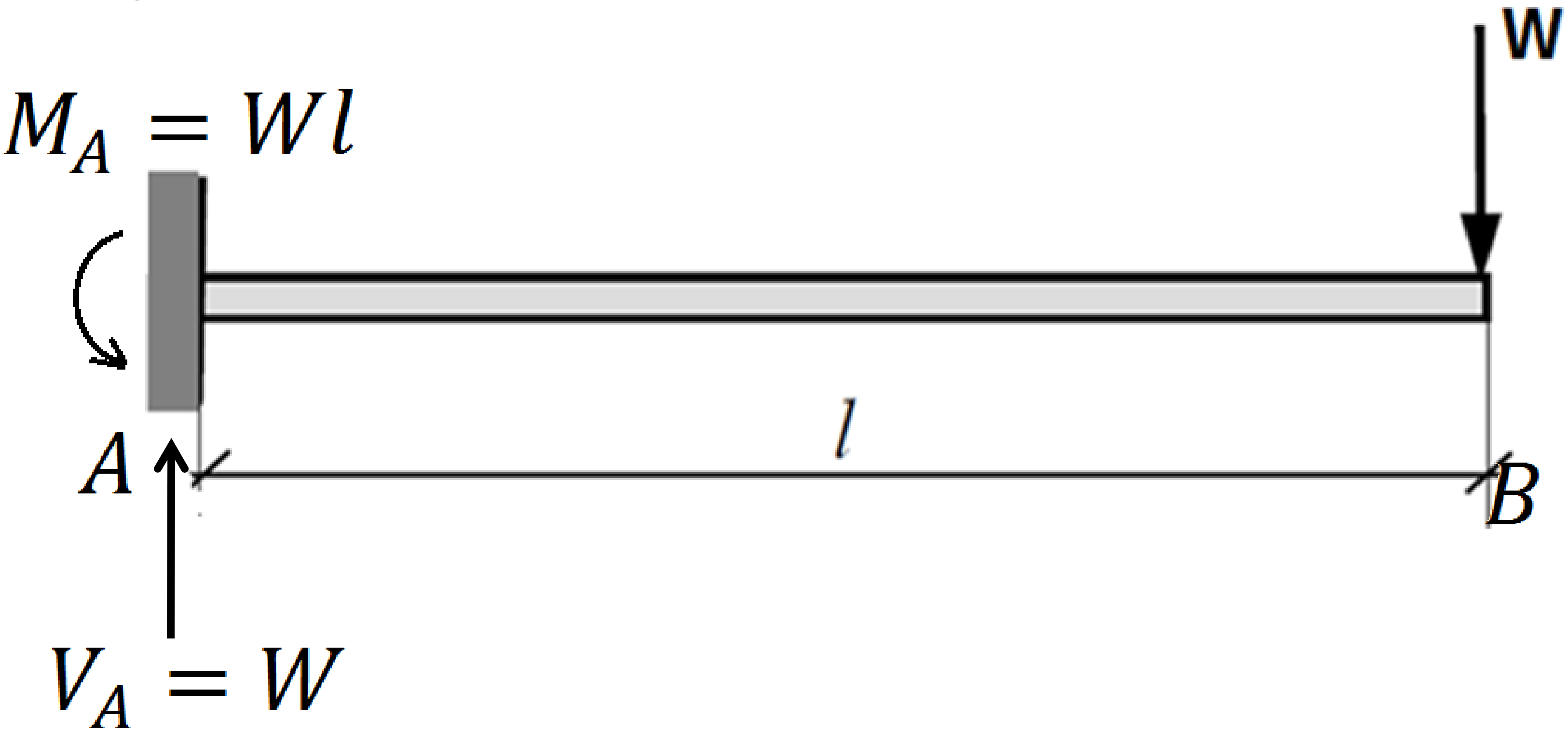

A
2 m
18 kNm
4 m
34 kNm
2 m
B
2 kN
2 kN


(-)
2 kN
SFD
2 kN


4 kNm
(+)
0
4 kNm
(-)
22 kNm
30 kNm
BMD

EX:22 Draw the shear force and bending moment diagram for a cantilever subjected to a point load at the free end.



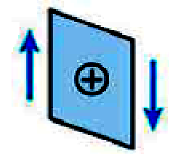

W
$M_A = Wl$
A
l
B
$V_A = W$

Solution:

Step 1: Support Reactions:

$V_A = W$; $M_A = Wl$

Step 2: Shear force diagram (SFD)

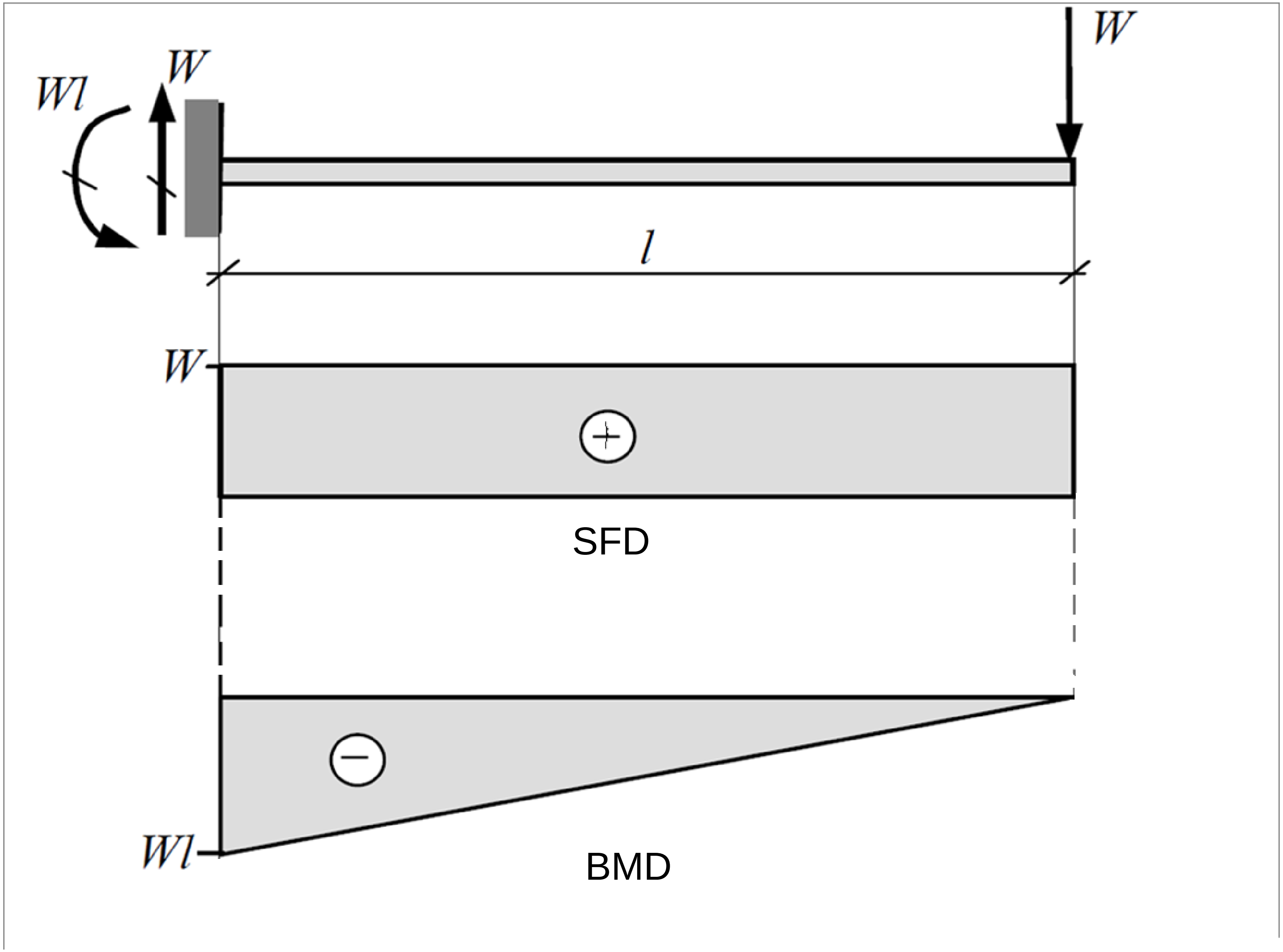


$(F_A)_L = 0$, $(F_A)_R = W$

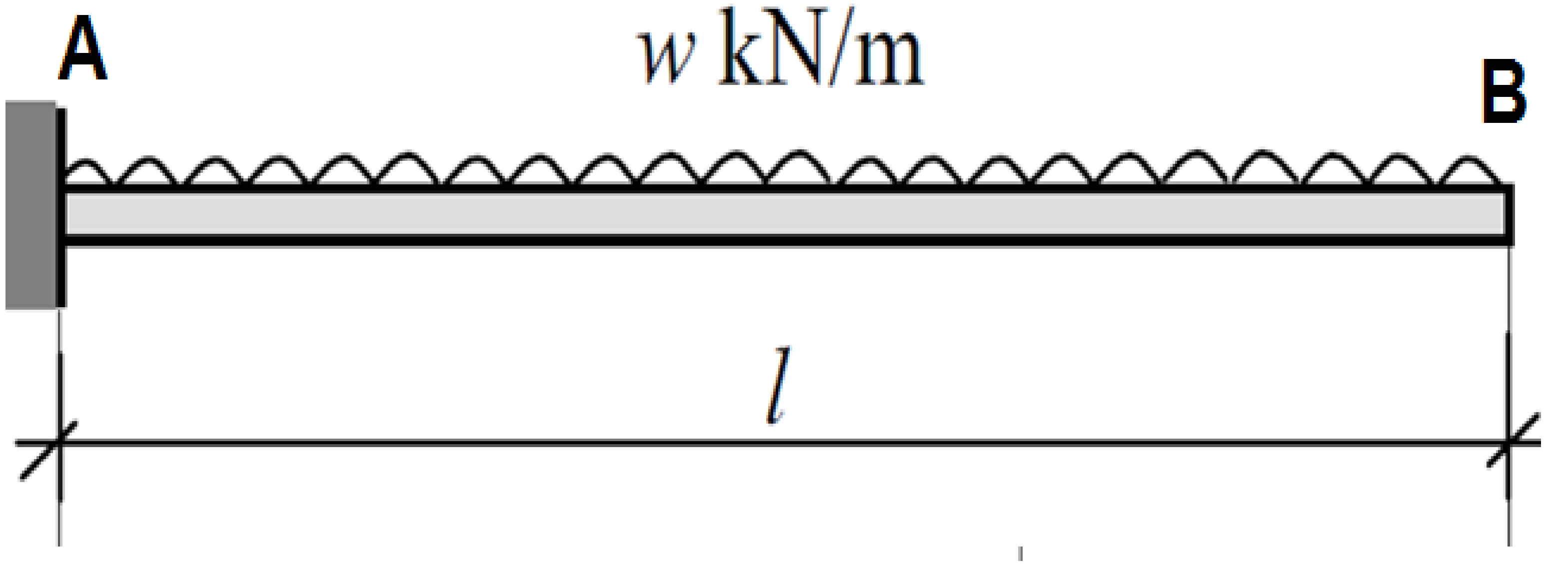
$(F_B)_L = -W$, $(F_B)_R = 0$

Step 3: Bending Moment Diagram (BMD)

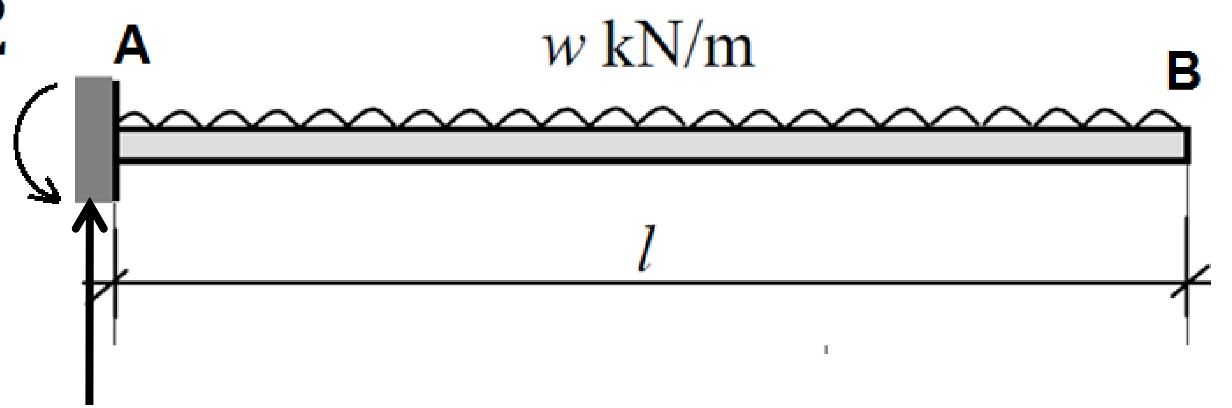
$(M_A)_L = 0$, $(M_A)_R = -Wl$, $M_B = 0$


W
Wl
W
l
W
SFD
Wl
BMD

Ex:23.Draw the shear force and bending moment diagram for a cantilever subjected to a u.d.l over the entire span



$$M_A = \frac{wl^2}{2}$$



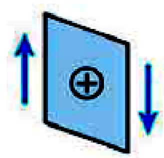
$$V_A = wl$$

Solution:

Step 1: Support Reactions:

$$V_A = wl \qquad\qquad M_A = \frac{wl^2}{2}$$

Step 2: Shear force diagram (SFD)

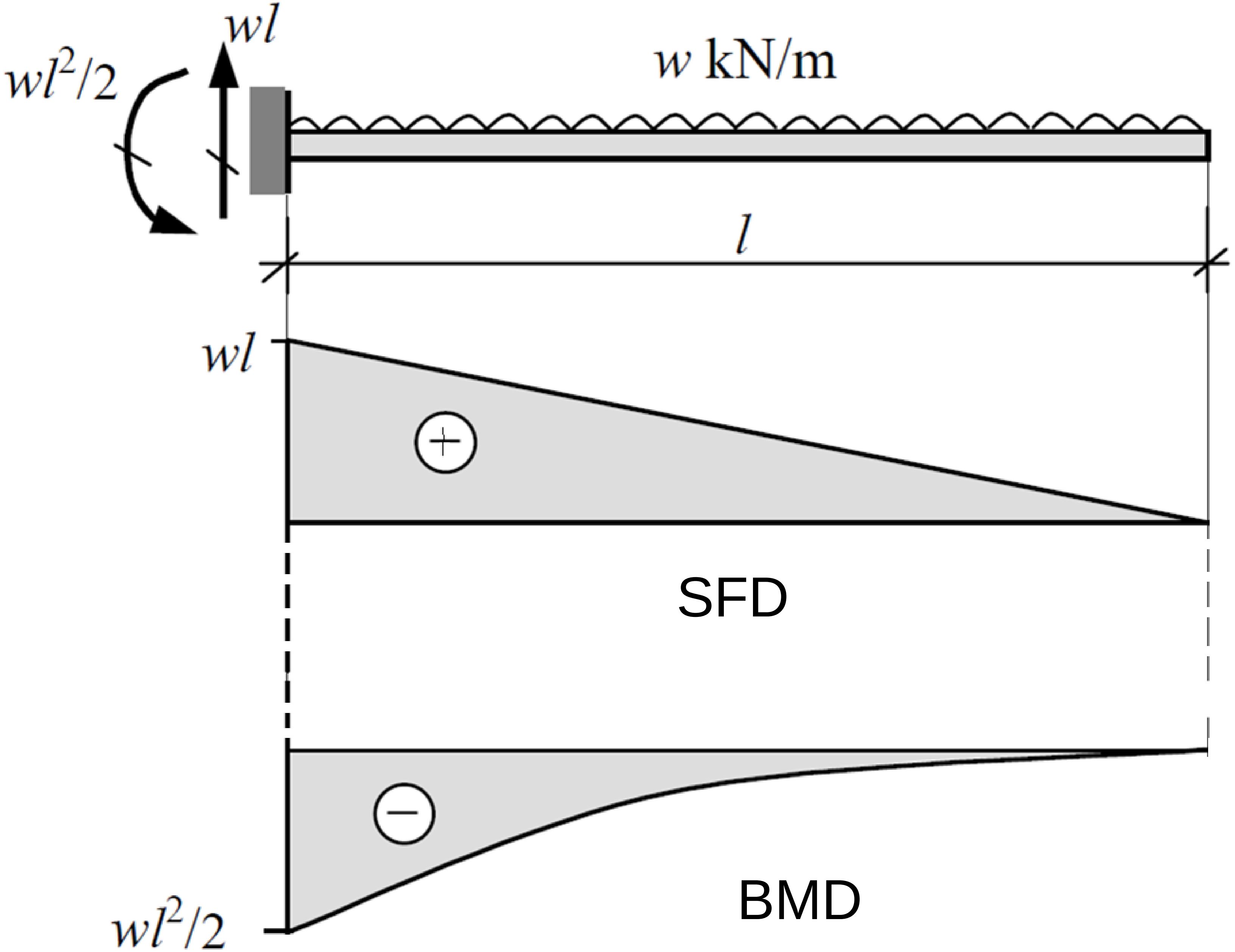


$(F_A)_L = 0, \qquad (F_A)_R = wl$

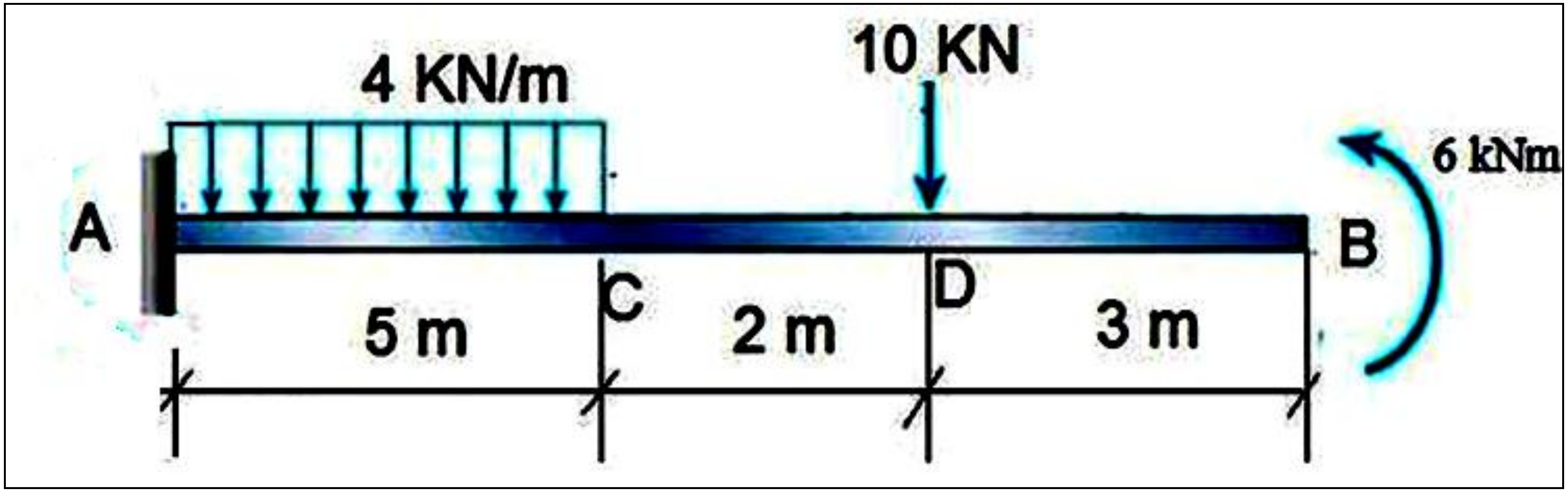
$F_B = 0$

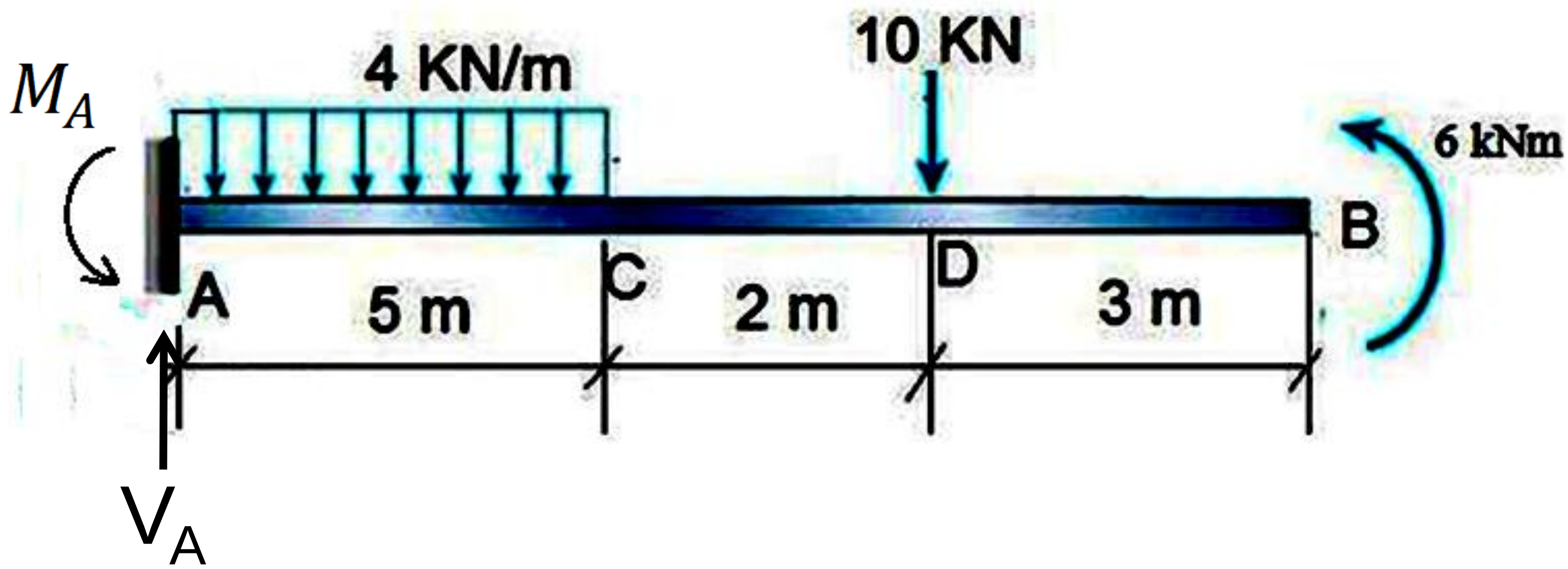
Step 3: Bending Moment Diagram (BMD)

$(M_A)_L = 0, \qquad (M_A)_R = -\dfrac{wl^2}{2}, \qquad M_B = 0$


wl
w kN/m
wl²/2
l
wl
SFD
BMD
wl²/2

Ex:24. Draw the shear force and bending moment diagram for a cantilever shown below





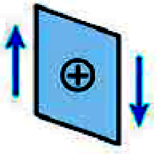
Soln:

Step 1: Support Reactions

$$V_A = 4 \times 5 + 10 = 30\ \text{kN} \uparrow$$

$$M_A = 4 \times 5 \times \frac{5}{2} + 10 \times 7 - 6 = 114\ kNm$$

<u>Step 2:</u> Shear force diagram (SFD)



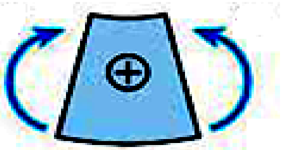
$(F_A)_L = 0,$ $(F_A)_R = 30$ kN

F_C =30-4X5=10 kN

$(F_D)_L = 10$kN, $(F_D)_R = 10\text{-}10=0$

F_B =0

Step 3: Bending Moment Diagram (BMD)



$$(M_B)_R = 0$$

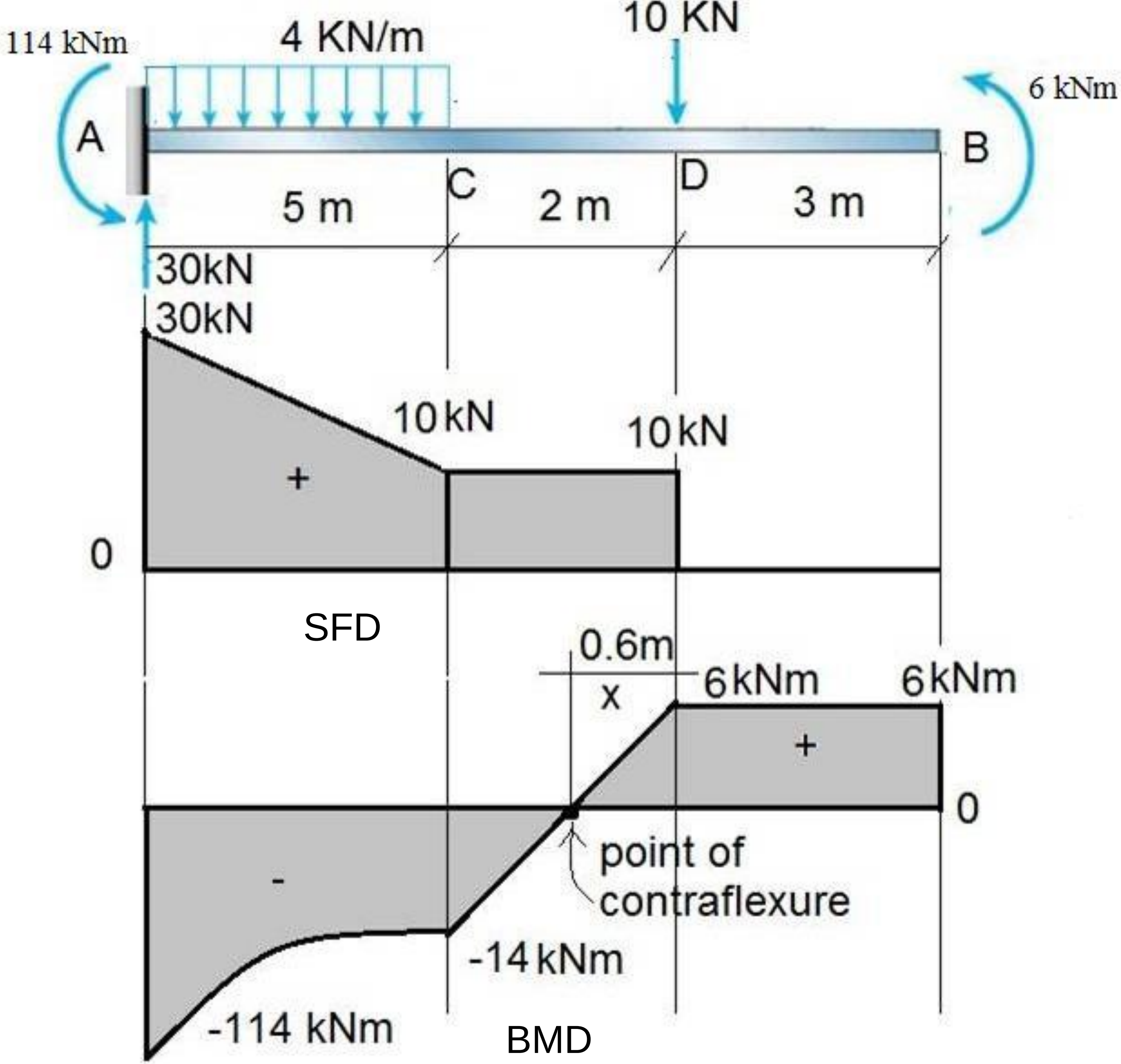
$$(M_B)_L = 6\ kNm$$

$$M_D = 6\ kNm$$

$$M_C = 6 - 10\ \times 2\ =\ -14\ kNm$$

$(M_A)_R$ = - 114 kNm

$(M_A)_L$ = 0



114 kNm
4 KN/m
10 KN
6 kNm
A
B
C
D
5 m
2 m
3 m
30kN
30kN
10 kN
10 kN
+
0
SFD
0.6m
x
6 kNm
6 kNm
+
0
-
point of
contraflexure
-14 kNm
-114 kNm
BMD

POINT OF CONTRAFLEXURE

From B.M diagram it is concluded that the point of contra flexure occurs in the portion DC, at a distance x from D.

- From similar triangle principle

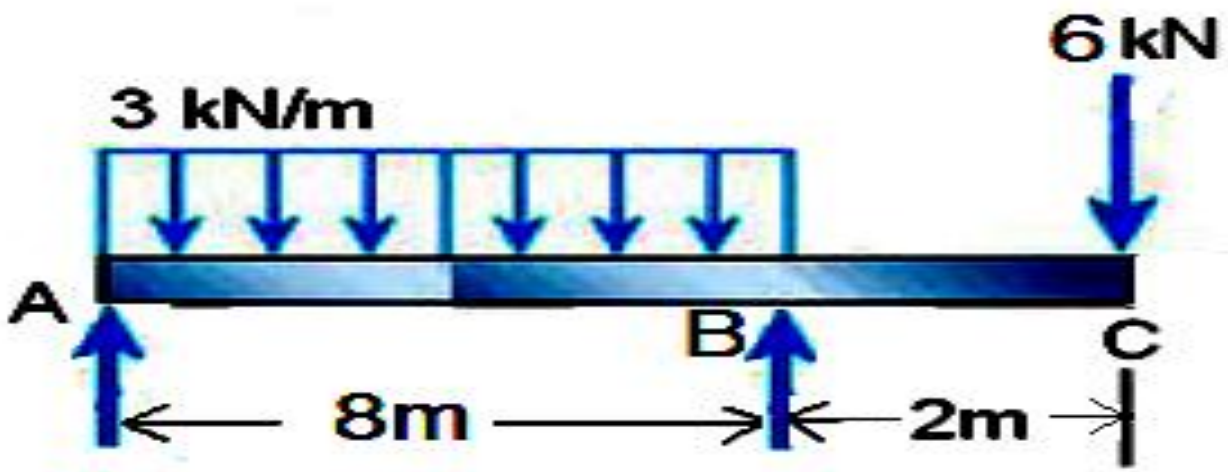
$$\frac{6}{x} = \frac{14}{2-x}$$

$$i.e., \quad \frac{6}{x} = \frac{20}{2}$$

Therefore, x= 0.6 m

OVERHANGING BEAMS

Ex:25. Draw the shear force and bending moment diagram for the beam shown below



Step 1 Determination of support reactions

To find V_B , $\sum M_A = 0$

$$3 \times 8 \times 8/2 - V_B \times 8 + 6 \times 10 = 0$$

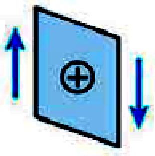
$$V_B = 19.5 \text{ kN}$$

To find V_A , $\sum V = 0$

$$V_A - 3 \times 8 + V_B - 6 = 0$$

$$V_A = 10.5 \text{ kN}$$

Step 2 shear force diagram



$$(F_A)_L = 0 \text{ kN}$$

$$(F_A)_R = 10.5 \text{ kN}$$

$$(F_B)_L = 10.5 – 24 = -13.5 \text{ kN}$$

$$(F_B)_R = 6 \text{ kN}$$

$$(F_C)_L = 6 \text{ kN}$$

$$(F_C)_R = 0 \text{ kN}$$

Step 3 Bending Moment Diagram

$M_A = 0$,

$M_B = -6 \times 2 = -12$ kNm

and $M_C = 0$

Maximum BM

Occurs in the span AB at the point of zero shear

The shear force is zero at 10.5/3 = 3.5m from A

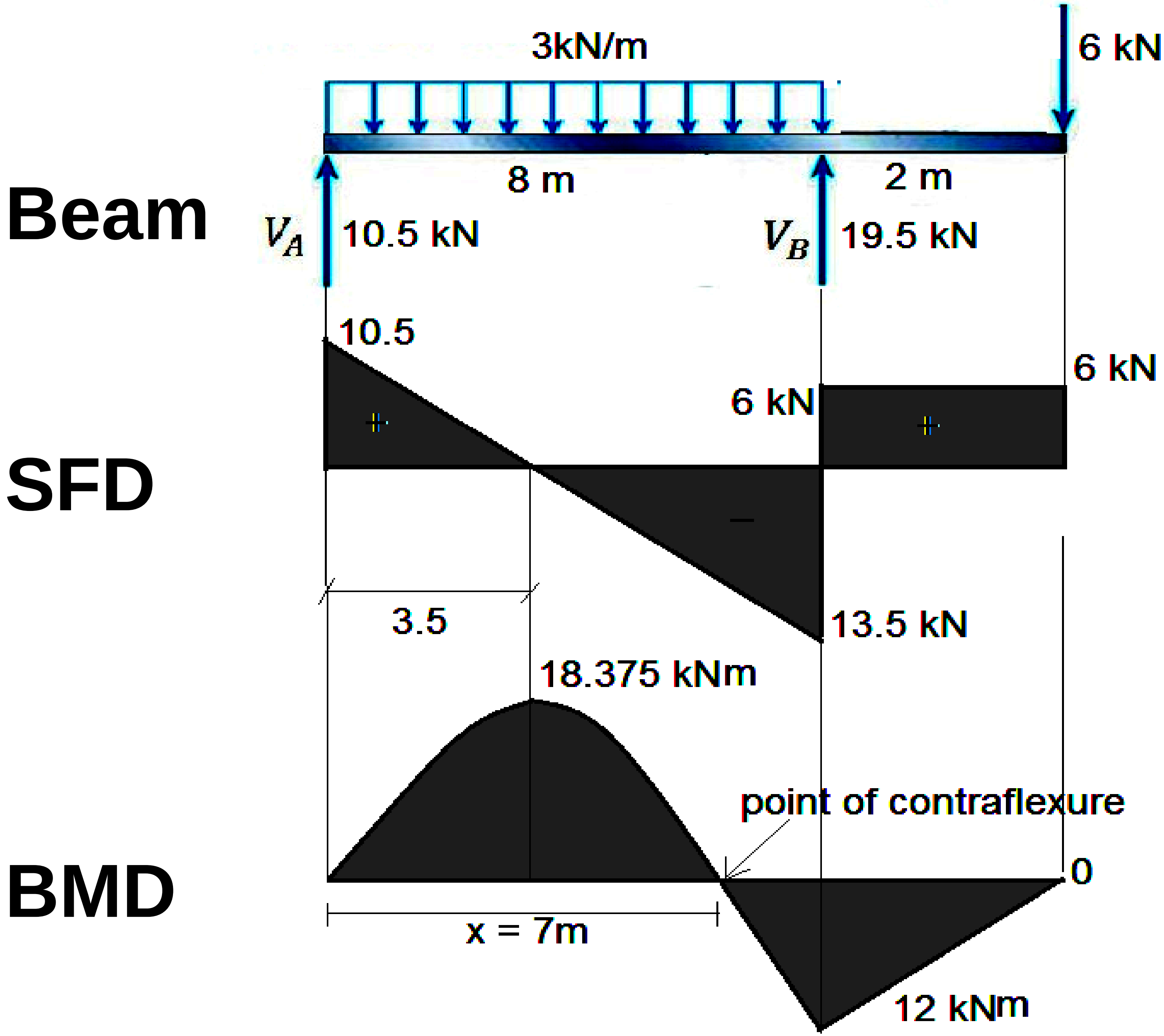
$M_{max} = 10.5 \times 3.5 - 3 \times 3.5^2 / 2 = 18.375$ kNm

Point of contra flexure :

The point of contra flexure occurs in the portion AB, at a distance x from A.

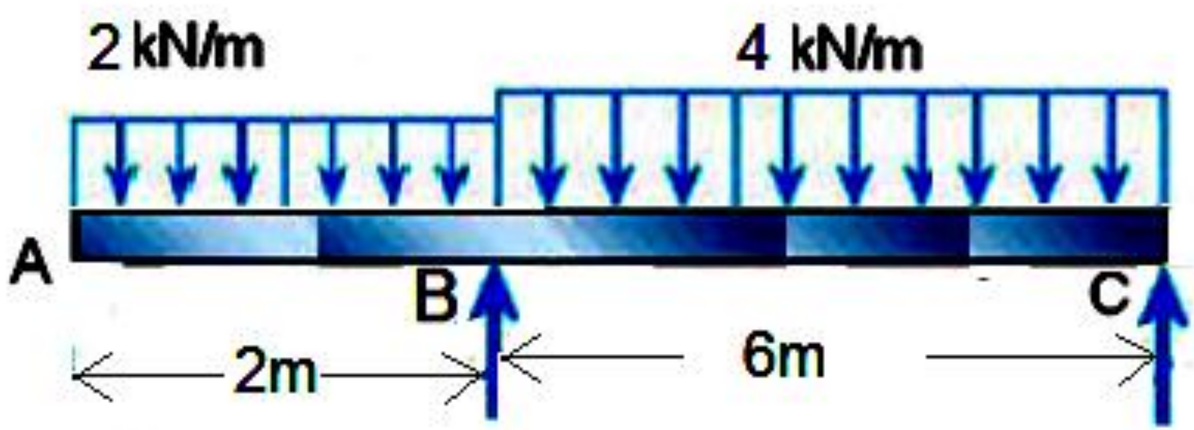
Therefore, $10.5x - \frac{3x^2}{2} = 0$

gives x = 0, 7m



Dr.K.Chinnaraju, Associate Professor, Division of Structural Engineering, Anna University, Chennai - 600 025.

Ex:26. Draw the shear force and bending moment diagram for the beam shown below



Step 1 Determination of support reactions

To find V_C , $\sum M_B = 0$ +

$$(-\ 2 \times 2 \times 1) + (4 \times 6 \times 3) - (V_C \times 6) = 0$$

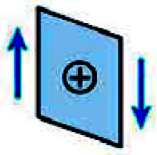
$$V_C = 11.33 \text{ kN}$$

To find V_B , $\sum V = 0$ +

$$(-2 \times 2) + V_B - (4 \times 6) + V_C = 0$$

$$V_B = 16.67 \text{ kN}$$

Step 2 shear force diagram



$$(F_A) = 0 \text{ kN}$$

$$(F_B)_L = -2 \times 2 = -4 \text{ kN}$$

$$(F_B)_R = -4 + 16.67 = 12.67 \text{ kN}$$

$$(F_C)_L = -11.33 \text{ kN}$$

$$(F_C)_R = 0 \text{ kN}$$

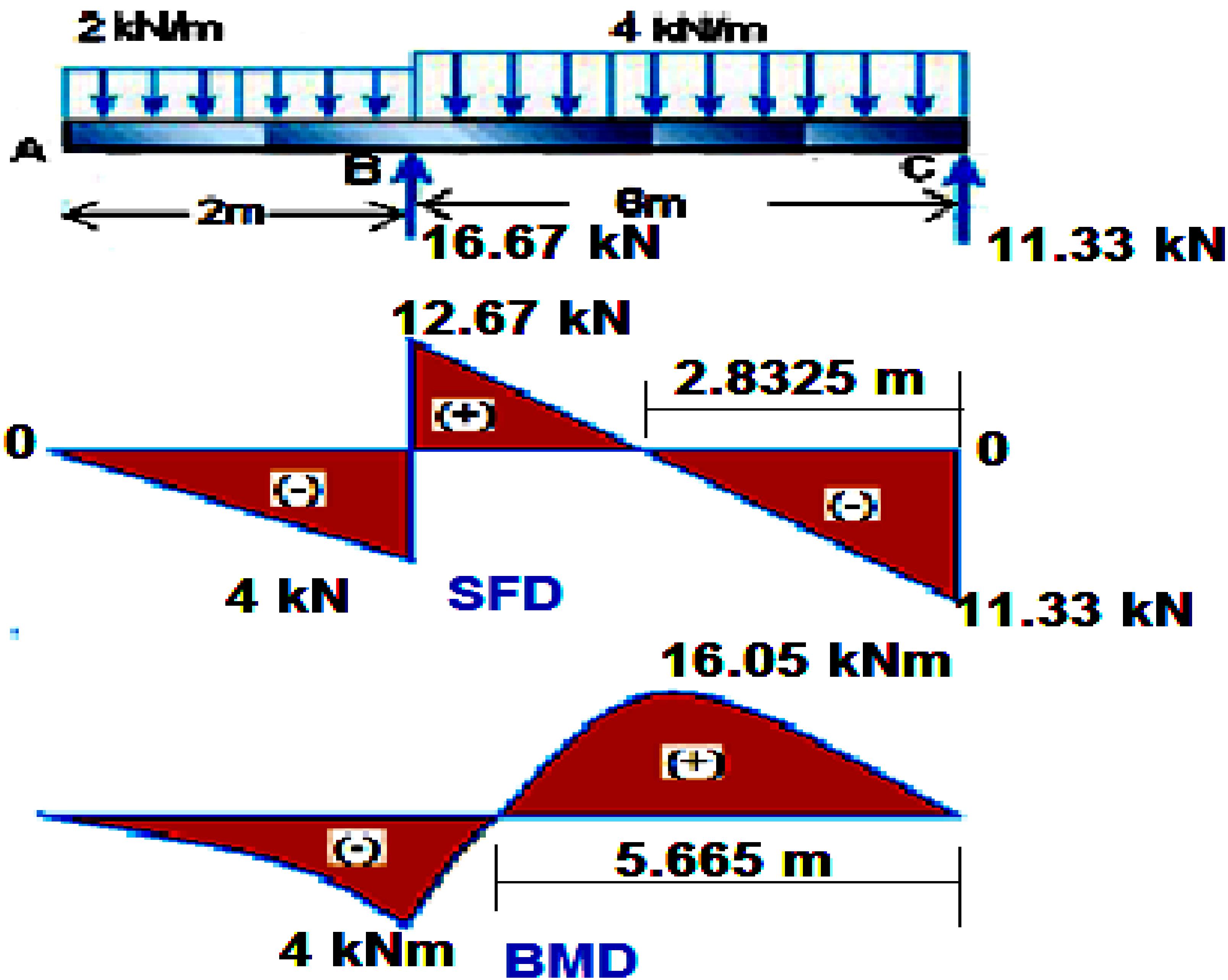
Step 3 Bending Moment Diagram

$M_A = 0$

$M_B = - 2 \text{ x } 2 \text{ x } 1 = - 4 \text{ kNm}$

$M_C = - 2 \times 2 \times 7 + 16.67 \times 6 - 4 \times 6 \times 3$

$= 0$


2 kN/m
4 kN/m
A
B
C
2m
6m
16.67 kN
11.33 kN
12.67 kN
2.8325 m
0
(+)
(-)
0
(-)
4 kN
SFD
11.33 kN
16.05 kNm
(+)
(-)
5.665 m
4 kNm
BMD

Maxmium Bending Moment

Position of zero shear(max BM):

X = 11.33/4 = 2.8325 m

Bending moment at a section x from C is given by $M_x = 11.33(x) – 2(x)^2$

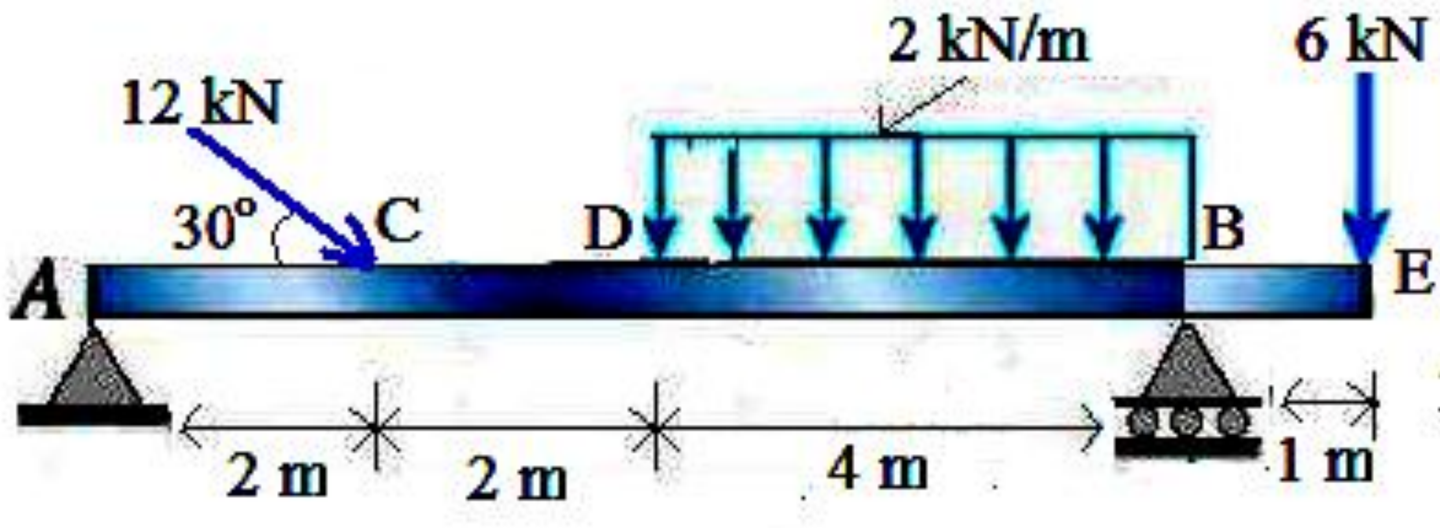
Therefore $M_{max} = 11.33(2.8325) – 2(2.8325)^2$

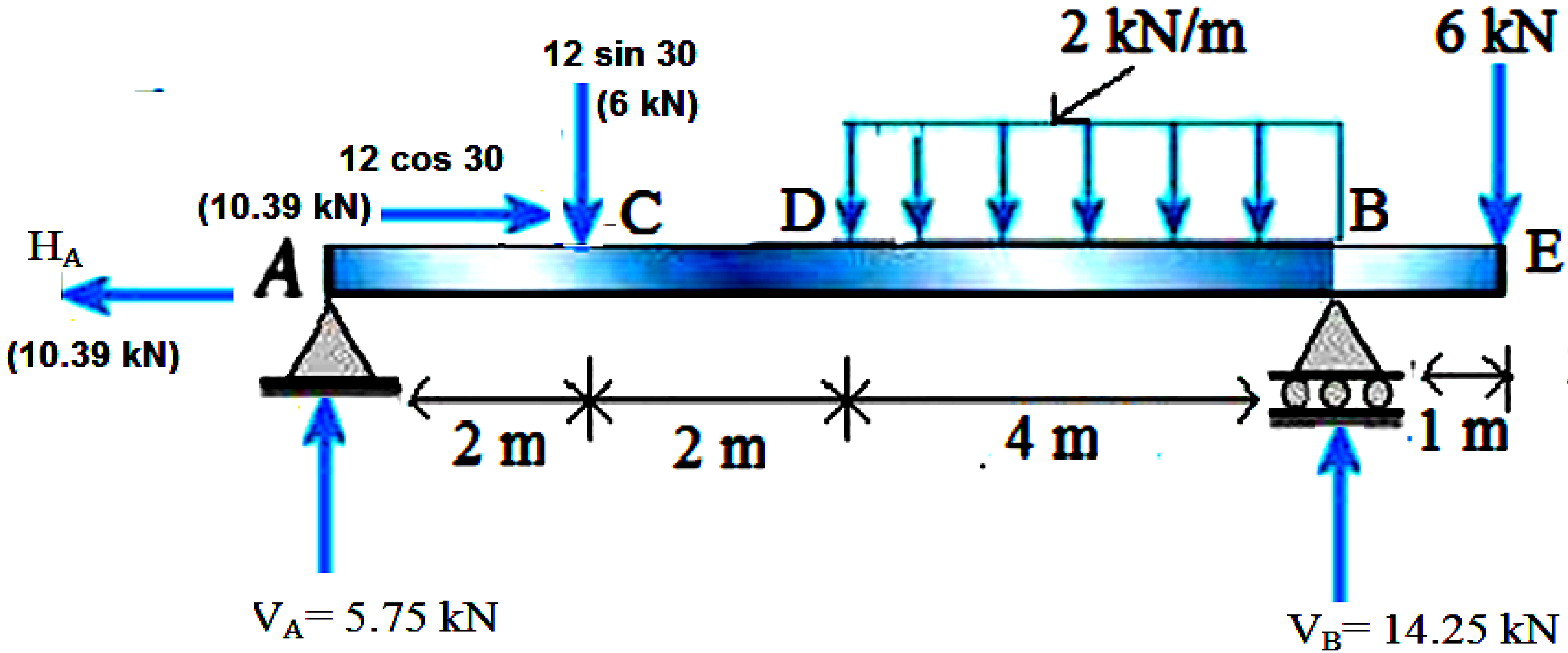
$= 16.05$ kNm

Point of contra flexure

$11.33(x) – 2(x)^2 = 0$, gives x = 0 , 5.665m

Ex.27 Analyse the beam shown in fig and draw the SFD and BMD. Mark all salient values and also draw the axial force diagram.





12 sin 30
(6 kN)
12 cos 30
(10.39 kN)
2 kN/m
6 kN
H_A
A
C
D
B
E
(10.39 kN)
2 m
2 m
4 m
1 m
V_A = 5.75 kN
V_B = 14.25 kN

Step 1 Determination of support reactions

$$H_A = 12\cos(30) = 10.39 \text{ kN} \leftarrow$$

To find V_B , apply $\sum M_A = 0$ $+\circlearrowright$

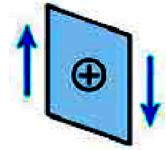
$$(6 \times 2) + (2 \times 4 \times 6) + (6 \times 9) - (V_B \times 8) = 0$$

$$V_B = 14.25 \text{ kN}$$

$$V_A = (6+8+6) - 14.25$$

$$= 5.75 \text{ kN}$$

Step 2 shear force diagram



$$(F_A)_L = 0$$

$$(F_A)_R = 5.75\ kN$$

$$(F_B)_L = 6 - 14.25$$
$$= -8.25\ kN$$

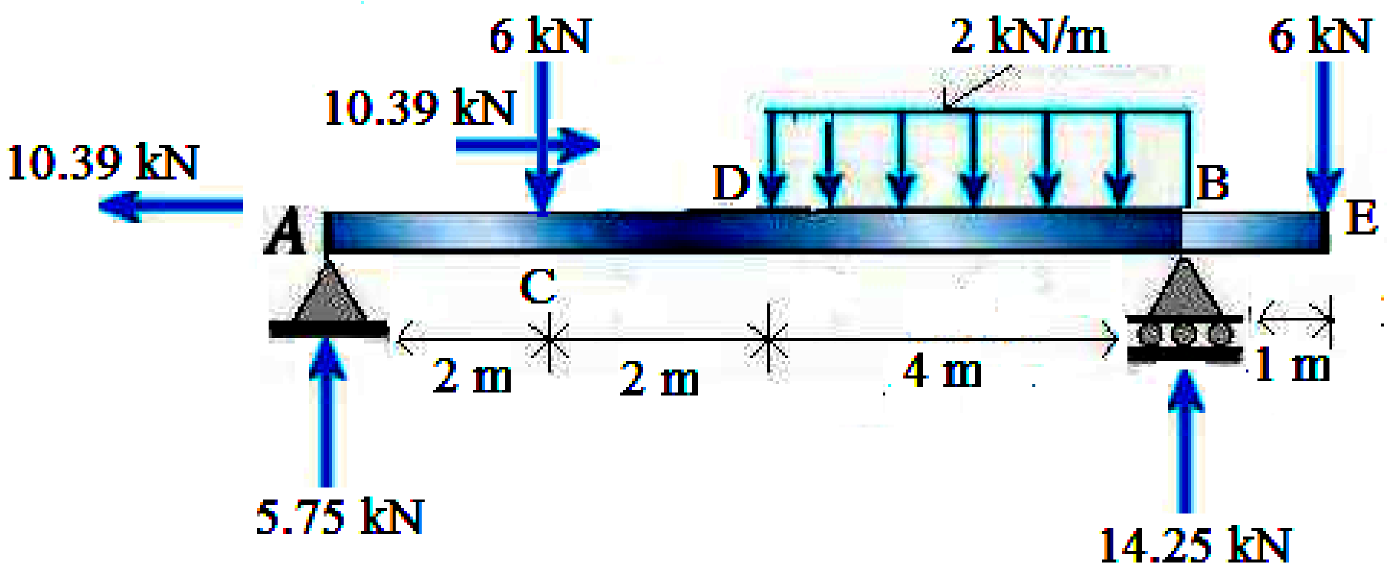
$$(F_B)_R = 6\ kN$$

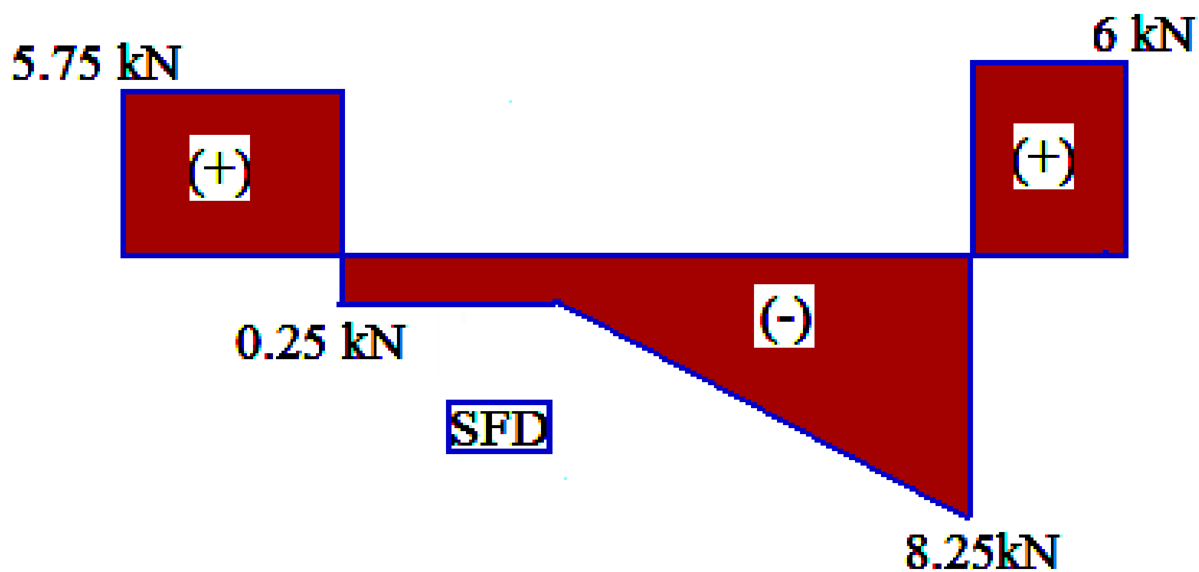
$$(F_C)_L = 5.75\ kN$$

$$(F_C)_R = 5.75 - 6$$
$$= -0.25 kN$$

$$(F_E)_L = 6\ kN$$

$$(F_E)_R = 0$$


6 kN
2 kN/m
6 kN
10.39 kN
10.39 kN
D
B
A
E
C
2 m
2 m
4 m
1 m
5.75 kN
14.25 kN


6 kN
5.75 kN
(+)
(+)
(-)
0.25 kN
SFD
8.25kN

Step 3 Bending Moment Diagram

$$M_A = 0$$

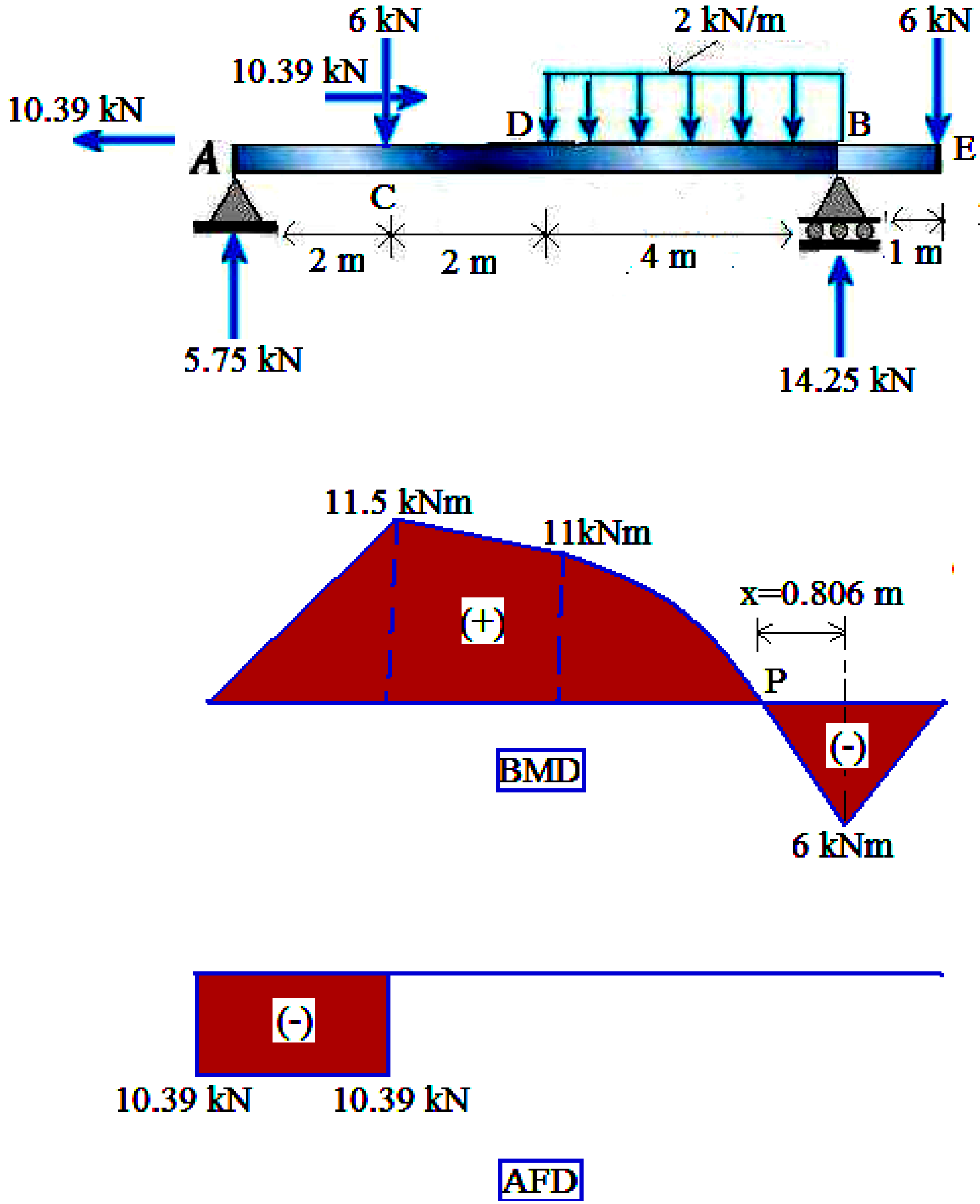
$$M_C = 5.75 \times 2 = 11.5\,kNm$$

$$M_D = 5.75 \times 4 - 6 \times 2$$

$$= 11\,kNm$$

$$M_B = -6 \times 1 = -6\,kNm$$

$$M_E = 0$$



6 kN
2 kN/m
6 kN
10.39 kN
10.39 kN
D
B
E
A
C
2 m
2 m
4 m
1 m
5.75 kN
14.25 kN
11.5 kNm
11kNm
x=0.806 m
(+)
P
(-)
BMD
6 kNm
(-)
10.39 kN
10.39 kN
AFD

Point of Contra flexure

Let the point of contra flexure (P) be at a distance x from B in the portion DB.

We know the BM at any section X in DB at a distance x from B,

$$M_x = 6(x+1) + 2\frac{x^2}{2} - 14.25x$$

$$= 6x + 6 + x^2 - 14.25\,x$$

$$= x^2 - 8.25x + 6$$

At P, BM = 0

X = 0.806 m , 7.44 m (which is not admissible)

$$\therefore x = 0.806\, m$$

<u>Axial Force Diagram</u>

Portion AC is subjected to an axial tensile force of 10.39 kN

$$(A_A)_L = 0$$

$$(A_A)_R = -10.39\ kN$$

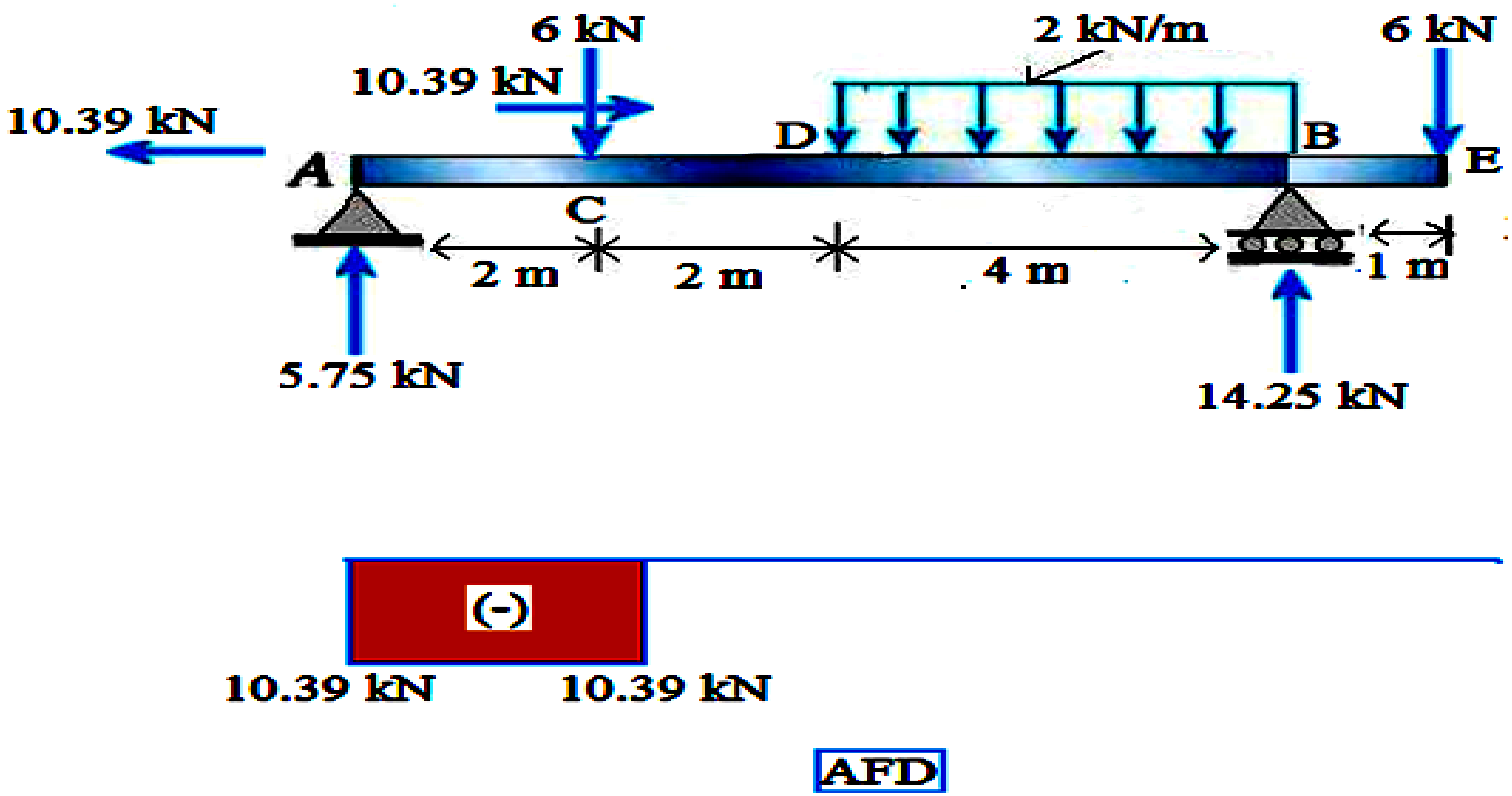
$$(A_C)_L = -10.39\ kN$$

$$(A_C)_R = 0$$

$$A_D = 0\ kN$$

$$A_B = 0$$

$$A_E = 0$$



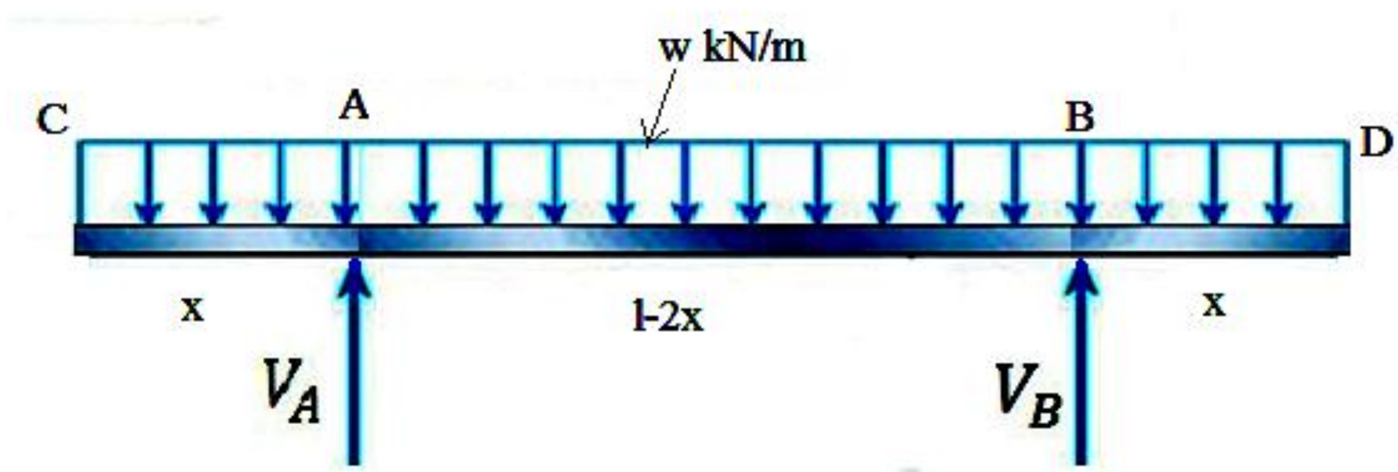
6 kN
10.39 kN
10.39 kN
2 kN/m
6 kN
D
B
E
A
C
2 m
2 m
4 m
1 m
5.75 kN
14.25 kN
(-)
10.39 kN
10.39 kN
AFD

Ex:28. A beam of length l metre carries a udl of w kN/m over the entire length and rests on two supports with both sides overhanging equally. At what fraction of the length must the supports be placed so that the maximum BM produced in the beam is least possible? Draw the SFD and BMD for the beam.

Soln:

Let the supports be placed at distance x from each edge of the beam.

For maximum BM to be least possible, magnitude of max –ve BM must be equal to magnitude of max +ve BM.



$$V_A = V_B = \frac{wl}{2}$$

$$\max -ve\ BM = \frac{-wx^2}{2}$$

Maximum +ve BM will occur at the centre

$$\max +ve\ BM = \frac{wl}{2}\left(\frac{l-2x}{2}\right) - \frac{wl}{2}\frac{l}{4}$$

$$= \frac{wl^2}{4} - \frac{wl}{2}x - \frac{wl^2}{8}$$

Therefore, $$\frac{wl^2}{8} - \frac{wl}{2}x = \frac{wx^2}{2}$$

$$l^2 - 4lx = 4x^2$$

$$4x^2 + 4lx + l^2 = 2l^2$$

$$(2x + l)^2 = 2l^2$$

$$2x + l = \pm\sqrt{2}\, l$$

$$2x = \pm\sqrt{2}\, l - l$$

$$x = \frac{\pm\sqrt{2}\, l - l}{2}$$

Negative value will not be admissible

$$\therefore\ x = \frac{\sqrt{2}\, l - l}{2} = 0.207\, l$$

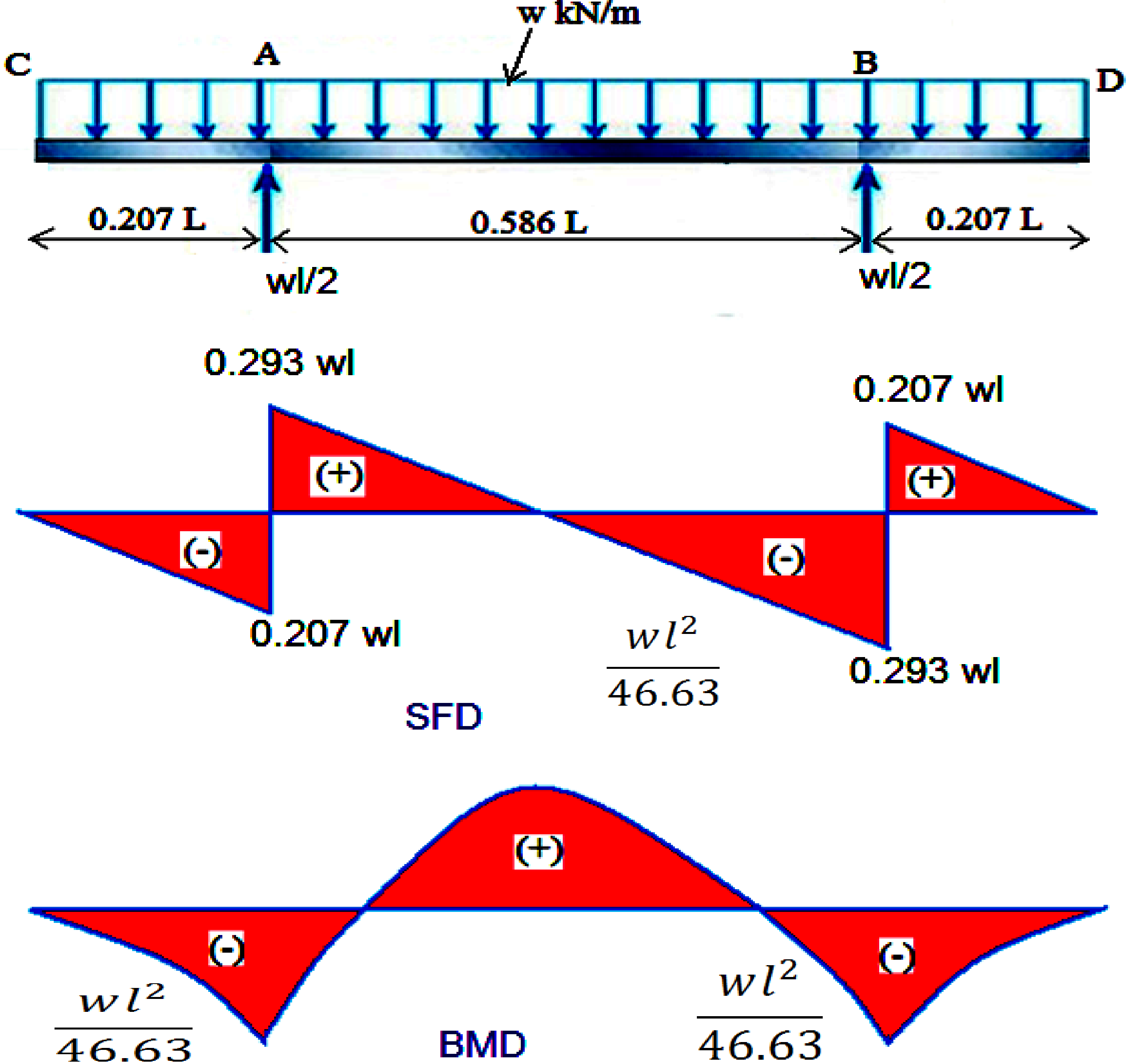
$$x = 0.207\, l$$

$$\max +ve\ BM = \max -ve\ BM = \frac{wx^2}{2}$$

$$= \frac{w(0.207l)^2}{2}$$

$$= 0.02145wl^2$$

$$= \frac{wl^2}{46.63}$$


w kN/m
C
A
B
D
0.207 L
0.586 L
0.207 L
wl/2
wl/2
0.293 wl
0.207 wl
(+)
(+)
(-)
(-)
0.207 wl
$\frac{wl^2}{46.63}$
0.293 wl
SFD
(+)
(-)
(-)
$\frac{wl^2}{46.63}$
$\frac{wl^2}{46.63}$
BMD

BENDING STRESSES IN BEAMS

- The bending moment at a section tends to bend or deflect the beam and the internal stresses resist its bending.
- The process of bending stops, when every cross section sets up full resistance to the bending moment.
- The resistance offered by the internal stresses, to the bending, is called bending stress, and the relevant theory is called the theory of simple bending.

Assumptions in Simple Bending Theory

1. The material of the beam is perfectly homogeneous (i.e., of the same kind throughout) and isotropic (i.e., of equal elastic properties in all directions).

2. The beam material is stressed within its elastic limit and thus, obeys Hooke's law.

3. The transverse sections, which were plane before bending, remain plane after bending also.

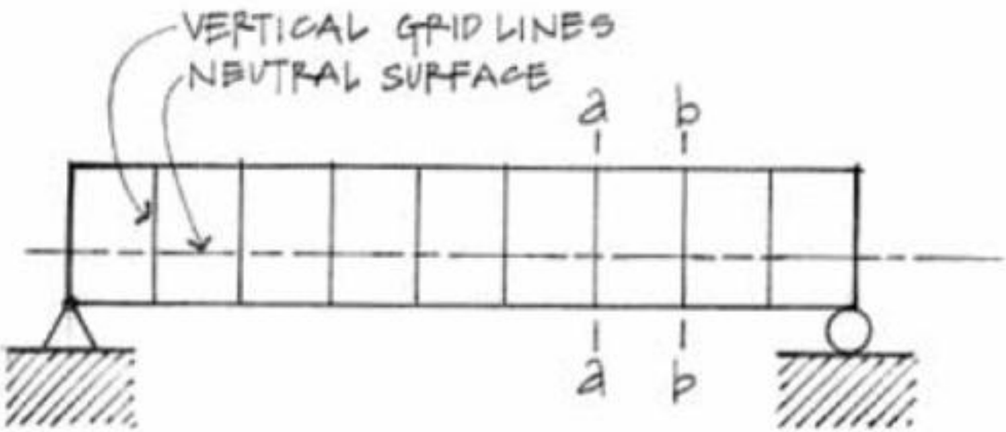
Assumptions in Simple Bending Theory

4. The beam is made up of number of layers placed one over the other, and each layer of the beam is free to expand or contract, independently, of the layer above or below it.

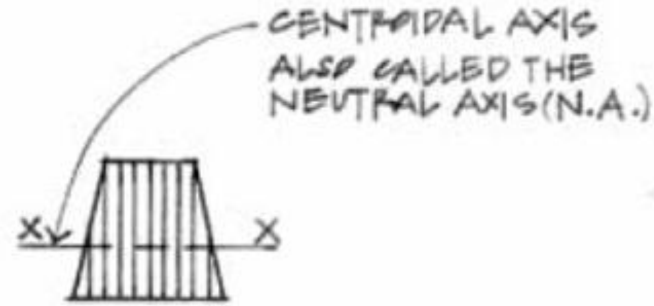
5. The value of modulus of elasticity is the same both in tension and compression.

6. The beam is in equilibrium i.e., there is no resultant pull or push in the beam section.

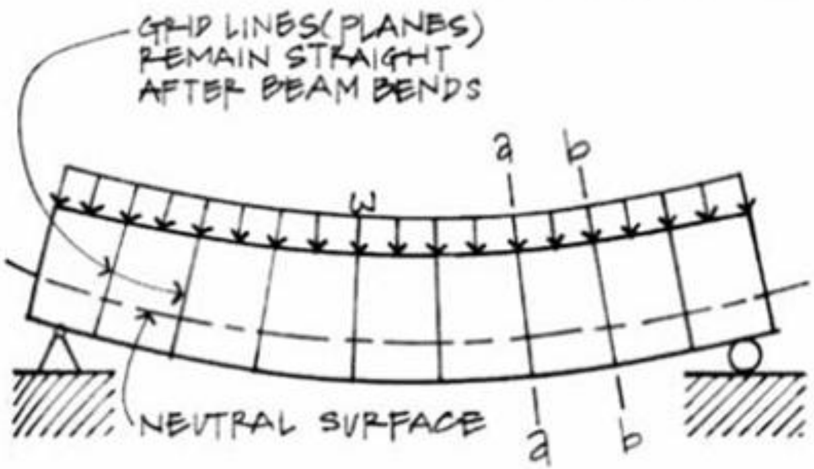
7. The beam is initially straight



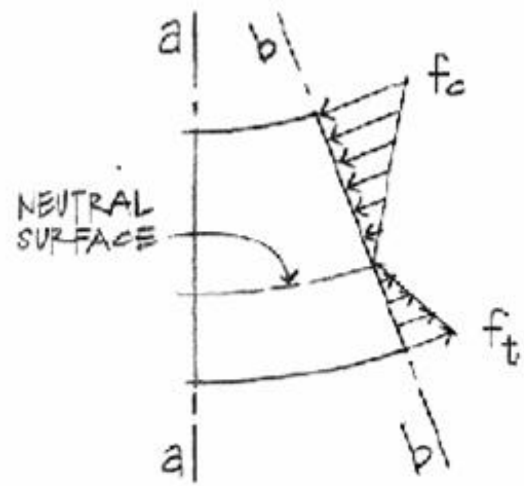
Beam elevation before loading



Beam cross section



Beam bending under load



Bending stresses on section b-b

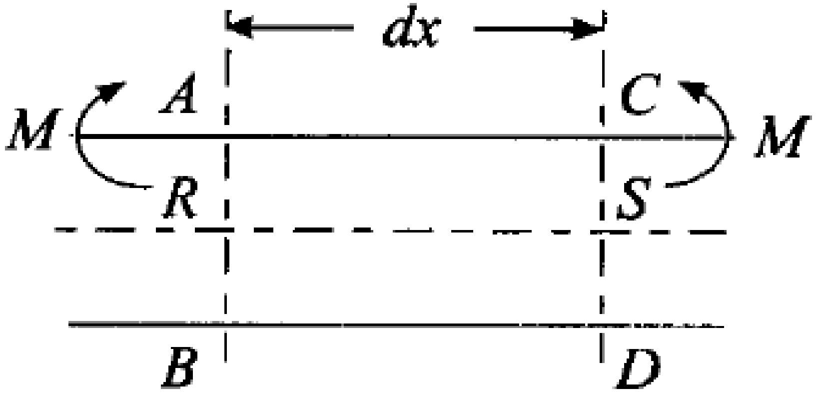
Theory of Simple Bending

- ❖Consider an initially straight beam, under pure bending.
- ❖The beam may be assumed to be composed of an infinite number of longitudinal fibers.
- ❖ Due to the bending, fibres in the lower part of the beam extend and those in the upper parts are shortened.
- ❖Somewhere in-between, there would be a layer or fibre that has undergone no extension or change in length.
- ❖This layer is called neutral layer.

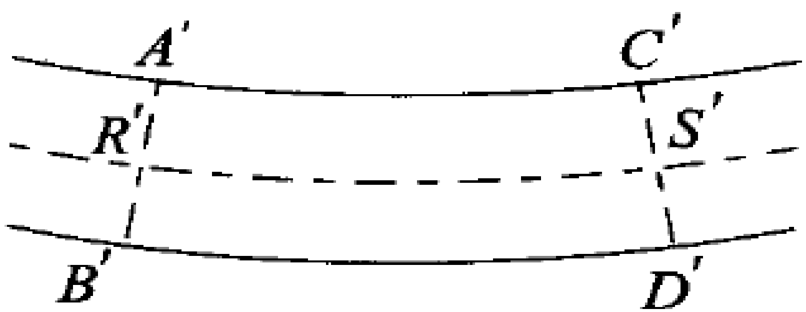
Theory of Simple Bending

- Consider a small length of a simply supported beam subjected to a bending moment as Shown in Fig. *(a).*
- *Now consider two sections AB and CD, which are normal to the axis of the beam RS.*
- Due to action of the bending moment, the beam as a whole will bend as shown in Fig. *(b).*

Theory of Simple Bending



(a) Before bending



(b) After bending

- **Bending Stress**
- Consider a small length dx of a beam subjected to a bending moment as shown in Fig.5.3

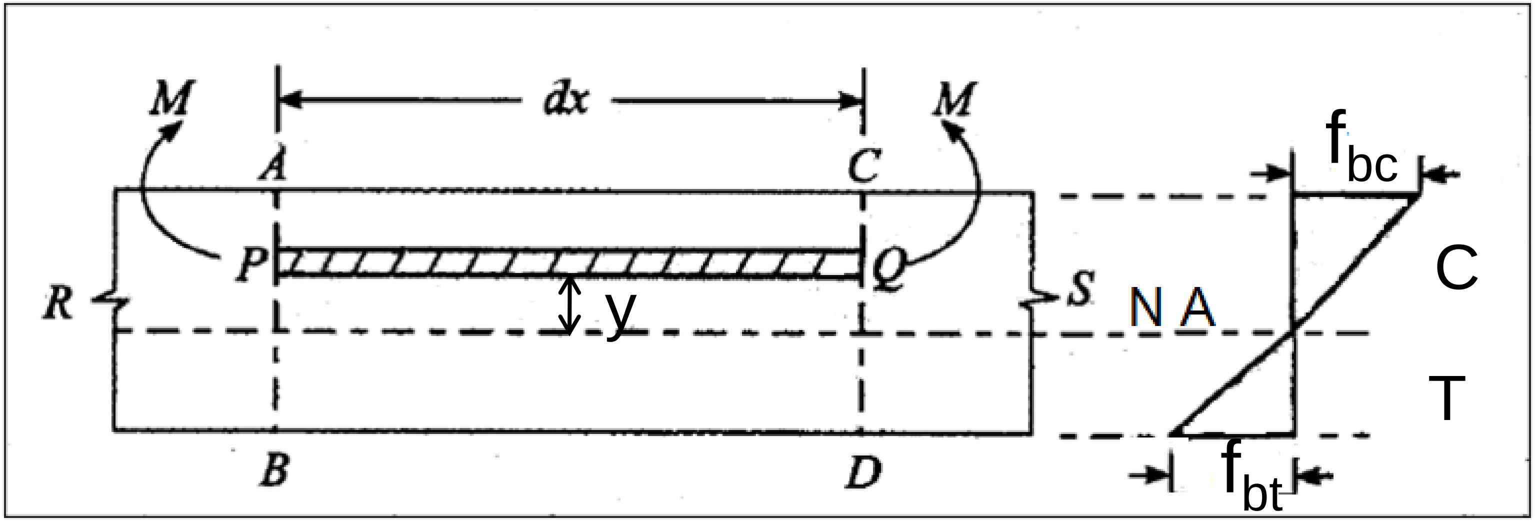
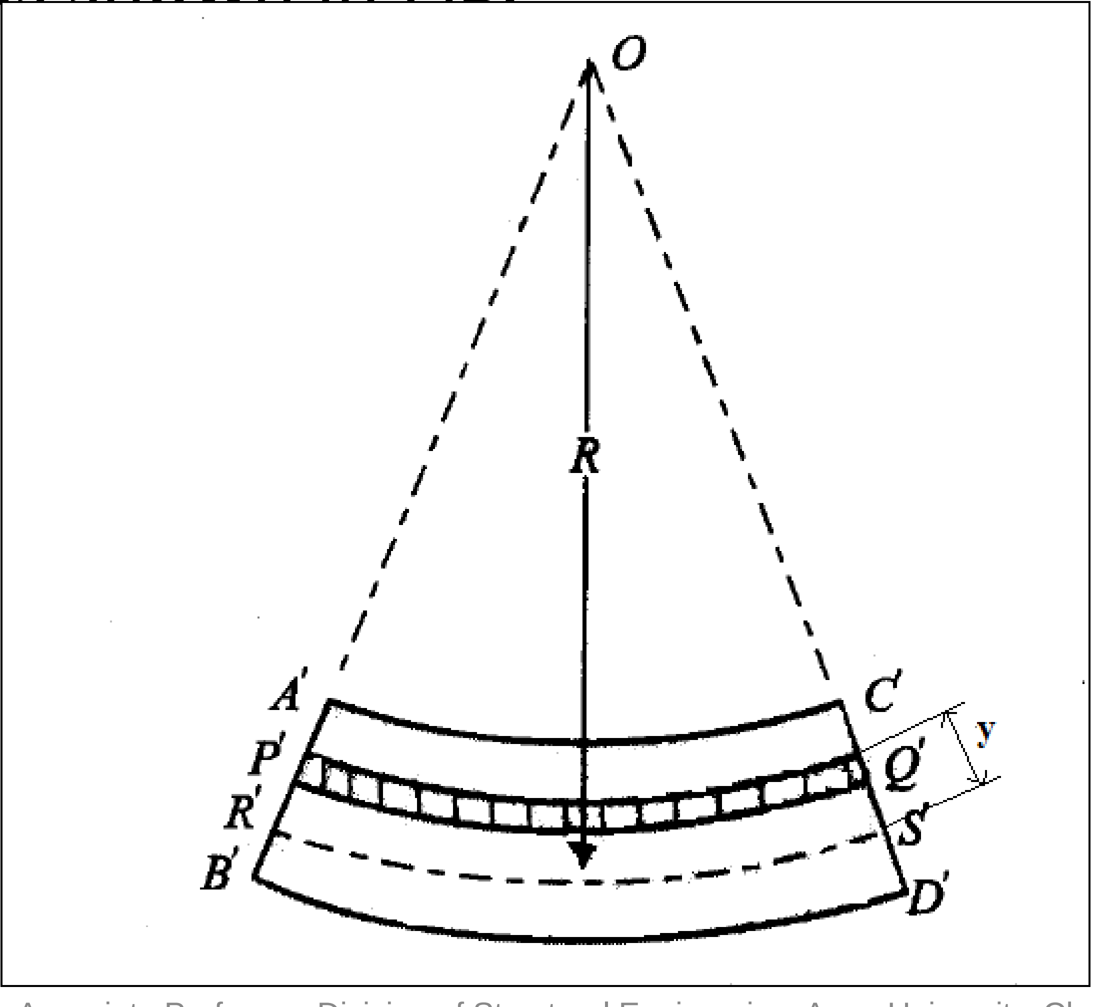


Fig.5.3

- As a result of this moment, let this small length of beam bend into an arc of a circle with 'O' as centre as shown in Fig.



- Now consider a layer *PQ at a distance 'y' from RS (the neutral axis of the beam).*
- *Let this layer be* compressed to *P'Q'(after bending)*
- Decrease in length of this layer,

$$\delta l = PQ - P'Q'$$

$$\therefore Strain = \frac{\delta l}{original\ length}$$

$$= \frac{PQ - P'Q'}{PQ}$$

- Now from the geometry of the curved beam (Fig.5.4), we find that the two sections $OP'Q'$ and $OR'S'$ are similar

$$\frac{P'Q'}{R'S'} = \frac{R-y}{R}$$

$$\therefore 1 - \frac{P'Q'}{R'S'} = 1 - \left(\frac{R-y}{R}\right)$$

$$\frac{R'S' - P'Q'}{PQ} = \frac{y}{R}$$

$$\frac{PQ - P'Q'}{PQ} = \frac{y}{R}$$

$$\varepsilon = \frac{y}{R}$$

- It is thus obvious, that the strain (ε) of a layer is proportional to its distance from the neutral axis
- We also know that the bending stress,

$$f_b = Strain \times Elasticity$$

$$= \varepsilon \times E$$

$$= \frac{y}{R} \times E$$

$$= y \times \frac{E}{R}$$

$$\therefore \frac{f}{y} = \frac{E}{R}$$

Position of Neutral Axis

δa = Area of the layer PQ.

Intensity of stress in the layer *PQ,*

$$f = y \times \frac{E}{R}$$

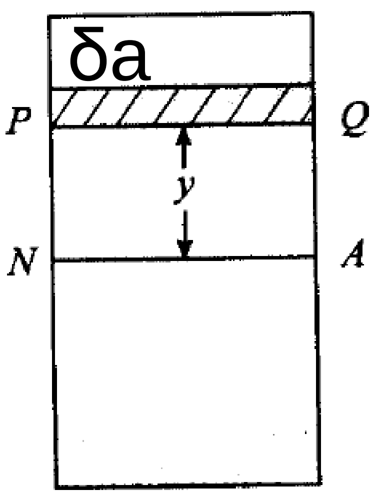
$\therefore$ Total force on the layer *PQ*

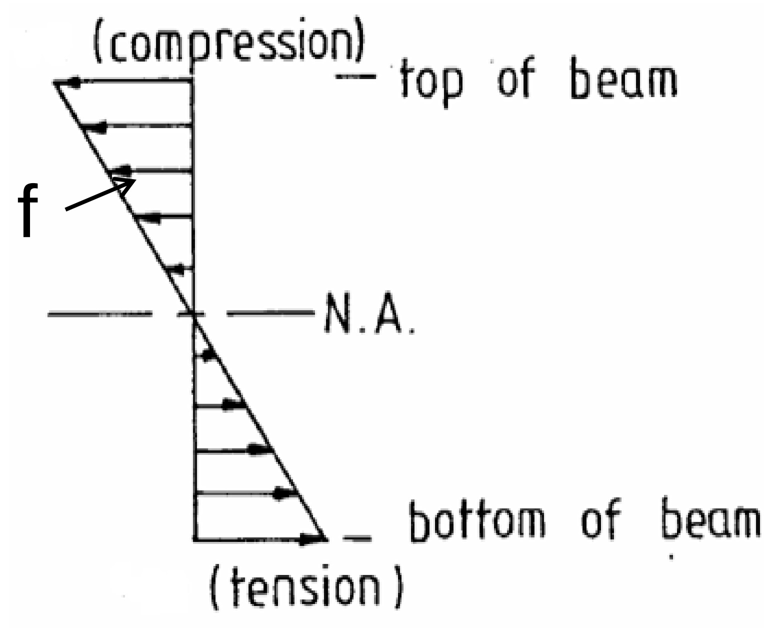
= Intensity of stress x Area

$$= y \times \frac{E}{R} \times \delta a$$

The intensity of stress in the layer *PQ* ,

$$f = y \times \frac{E}{R}$$





Total force on the section

$$= \sum y \times \frac{E}{R} \times \delta a = \frac{E}{R} \sum y.\delta a$$

Since the section is in equilibrium, therefore total force, from top to bottom, must be equal to zero.

$$\therefore \frac{E}{R} \sum y.\delta a = 0$$

$$\sum y.\delta a = 0$$

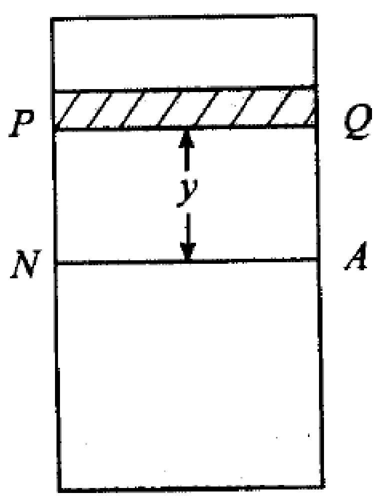
Thus, centroidal axis will be the neutral axis.

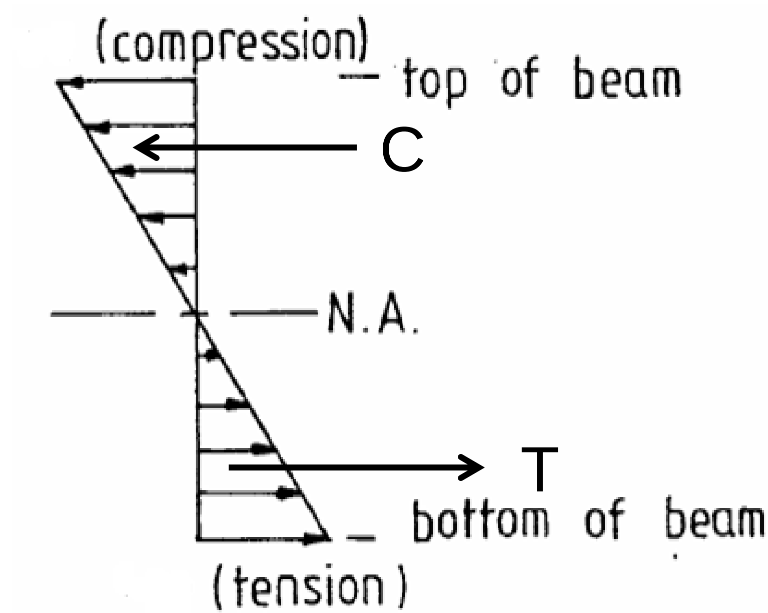
Moment of resistance

- we know, that on one side of the neutral axis there are compressive stresses and on the other there are tensile stresses.
- These stresses form a couple, whose moment must be equal to the external moment (M).
- The moment of this couple, which resists the external bending moment, is known as moment of resistance.

The intensity of stress in the layer *PQ* ,

$$f = y \times \frac{E}{R}$$





- Total force on the layer $PQ = y \times \frac{E}{R} \times \delta a$

and moment of this total force on the layer PQ about the neutral axis

$$= y \times \frac{E}{R} \times \delta a \times y = \frac{E}{R} y^2 . \delta a$$

The algebraic sum of all such moments about the neutral axis must be equal to M.

Therefore

$$M = \sum \frac{E}{R} y^2 . \delta a \quad = \frac{E}{R} \sum y^2 . \delta a$$

$$\therefore M = \frac{E}{R} \times I$$

$$\frac{M}{I} = \frac{E}{R}$$

- We already seen that, $\frac{f}{y} = \frac{E}{R}$

$$\therefore \frac{M}{I} = \frac{f}{y} = \frac{E}{R}$$

This is the theory of simple bending equation.

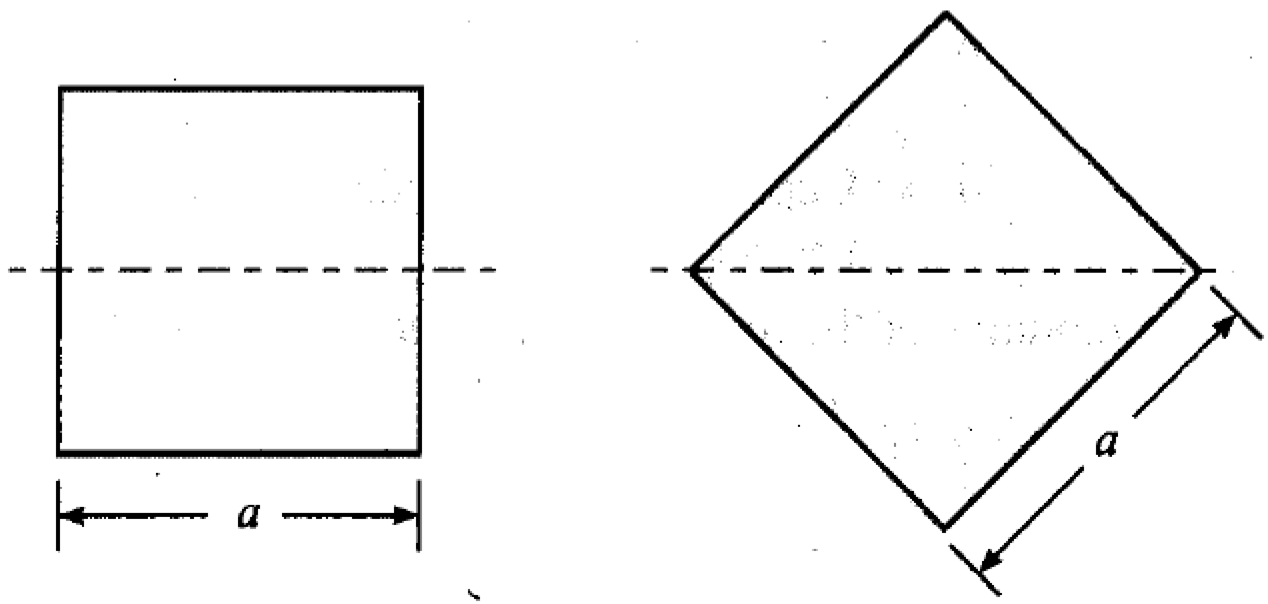
Strength of a Section

$$\frac{M}{I} = \frac{f}{y} \; or \; M = \frac{f}{y} I \; and \; M = fZ$$

- The moment of resistance depends upon second moment of area or section modulus of the section.
- The second moment of area of beam section does not depend upon its cross-section area, but its disposition in relation to the neutral axis.

- In the case of a beam, subjected to transverse loading, the bending stress at a point is directly proportional to its distance from the neutral axis.
- It is thus obvious that a larger area near the neutral axis of a beam is uneconomical. This idea is put into practice, by providing beams of section, where the flanges alone withstand almost all the bending stress.

Ex. 29. For a given stress, compare the moments of resistance of a beam of a square section, when placed (a) with its two sides horizontal and (b) with its diagonal horizontal as shown in Fig.



(a) (b)

Section modulus of the beam section with its two sides horizontal,

$$Z_1 = \frac{bd^2}{6} = \frac{a \times a^2}{6} = \frac{a^3}{6}$$

Second moment of area of the beam section with its diagonal horizontal,

$$I_2 = 2 \times \frac{bh^3}{12} = 2 \times \frac{a\sqrt{2}\left(\frac{a}{\sqrt{2}}\right)^3}{12} = \frac{a^4}{12}$$

and $$y_{max} = \frac{a}{\sqrt{2}}$$

$$\therefore Z_2 = \frac{I}{y_{max}} = \frac{\frac{a^4}{12}}{\frac{a}{\sqrt{2}}} = \frac{a^3}{6\sqrt{2}}$$

Since the moment of resistance of a section is directly proportional to their moduli of section, therefore,

$$\frac{M_1}{M_2} = \frac{Z_1}{Z_2} = \frac{\frac{a^3}{6}}{\frac{a^3}{6\sqrt{2}}} = \sqrt{2} = 1.414$$

Ex:30. *A rectangular beam is to be cut from a circular log of wood of diameter D. Find the ratio of dimensions for the strongest section in bending.*

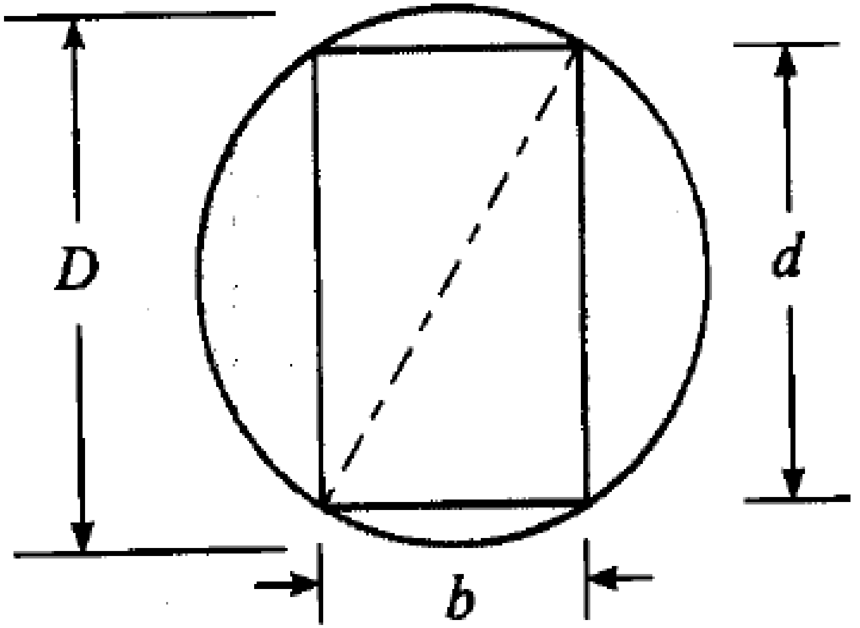


Fig.5.8

Section modulus of the rectangular section,

$$Z = \frac{bd^2}{6} = \frac{b \times (D^2 - b^2)}{6} = \frac{bD^2 - b^3}{6}$$

for Z to be maximum, $\frac{dZ}{db} = 0$

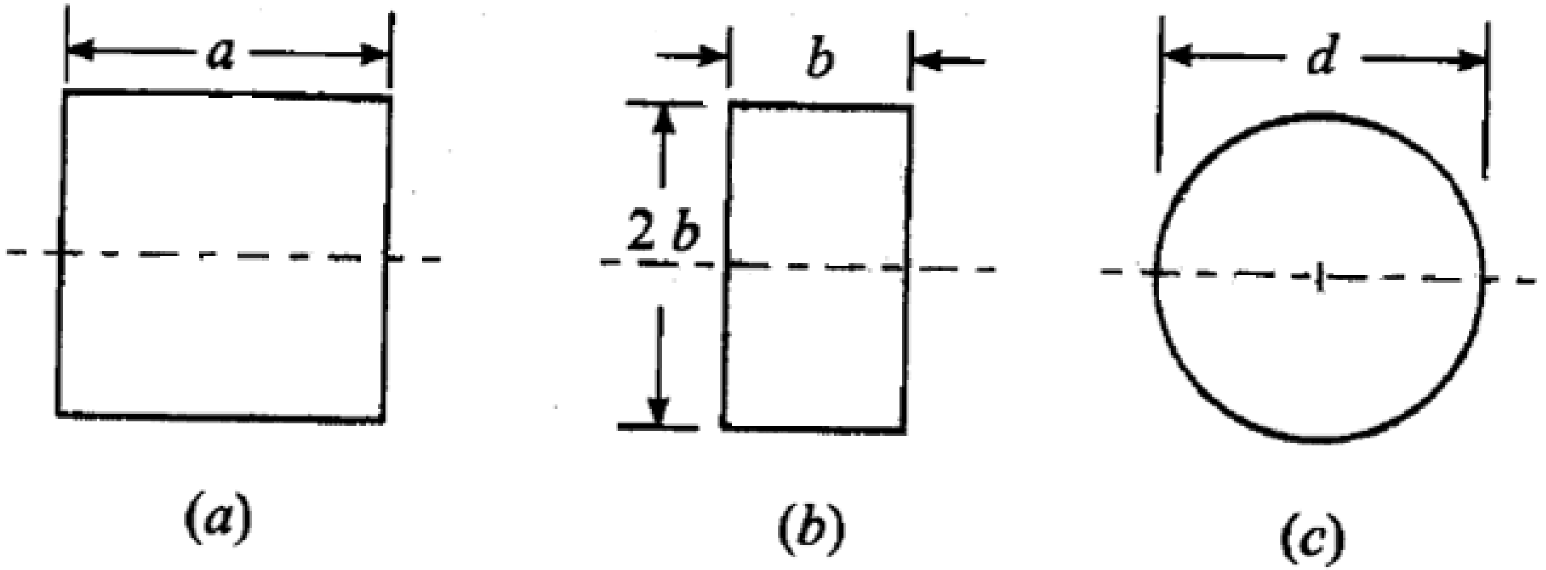
$$\frac{D^2 - 3b^2}{6} = 0 \quad or \quad D^2 - 3b^2 = 0$$

$$b = \frac{D}{\sqrt{3}} = 0.577D$$

$$and\ d^2 = D^2 - \frac{D^2}{3}$$

$$\therefore\ d = \sqrt{\frac{2}{3}}D = 0.8165D$$

Ex:31.Three beams have the same length, the same allowable stress and the same bending moment. The cross-section of the beams are a square, a rectangle with depth twice the width and a circle as shown in Fig. Find the ratios of weights of the circular and the rectangular beams with respect to the square beam.



- Since all the three beams have the same allowable stress (f) and bending moment *(M), therefore* the modulus of section of the three beams must be equal.
- We know that the section modulus for a square beam,

$$Z_1 = \frac{bd^2}{6} = \frac{a \times a^2}{6} = \frac{a^3}{6} \_\_\_\_\_(i)$$

Similarly, modulus of section for rectangular beam,

$$Z_2 = \frac{bd^2}{6} = \frac{b \times (2b)^2}{6} = \frac{2b^3}{3} \_\_\_\_\_(ii)$$

and modulus of section for a circular beam,

$$Z_3 = \frac{\pi}{32} \times d^3 \_\_\_\_\_ (iii)$$

Equating equations *(I) and (ii),*

$$\frac{a^3}{6} = \frac{2b^3}{3}$$

we get, ***b= 0.63a.***

equating equations *(i) and (iii),* $\frac{a^3}{6} = \frac{\pi}{32} \times d^3$

we get, ***d = 1.19a.***

$$\therefore \frac{weight\ of\ square\ beam}{weight\ of\ rectangular\ beam}$$

$$= \frac{area\ of\ square\ beam}{area\ of\ rectangular\ beam}$$

$$= \frac{a^2}{2b^2} = \frac{a^2}{2 \times (0.63a)^2} = \frac{1}{0.7937} = 1.26$$

∴Rectangular section is economical than square section

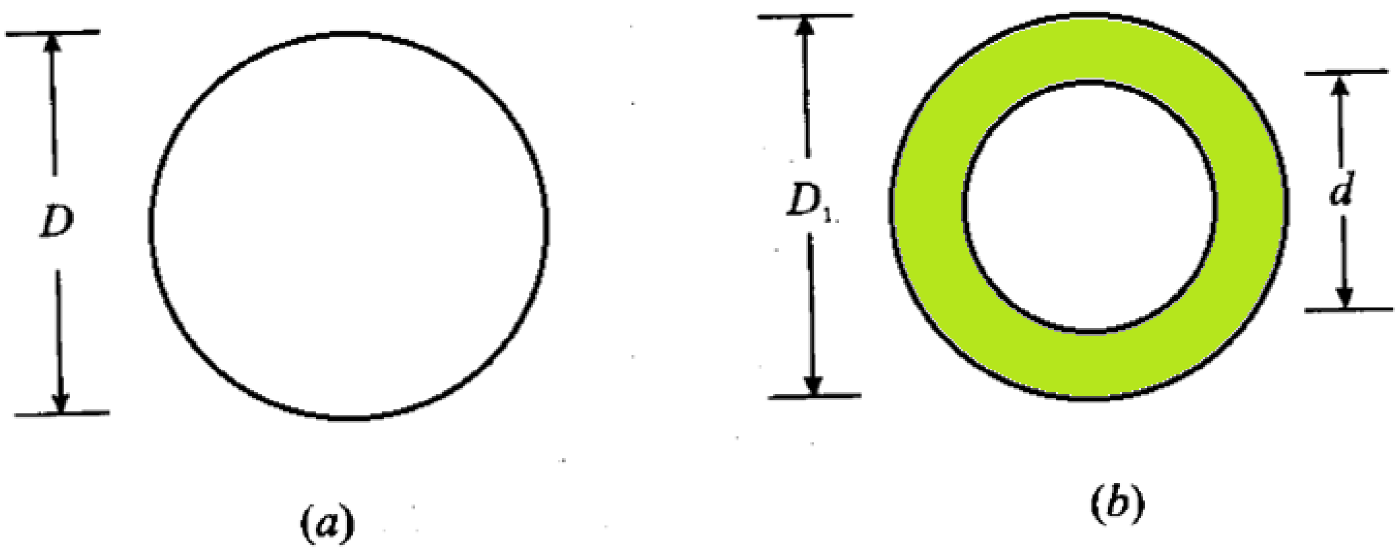
$$\frac{weight\ of\ square\ beam}{weight\ of\ circular\ beam} = \frac{area\ of\ square\ beam}{area\ of\ circular\ beam}$$

$$= \frac{a^2}{\frac{\pi}{4}\times d^2} = \frac{a^2}{\frac{\pi}{4}\times(1.19a)^2} = \frac{1}{1.12} = 0.893$$

$\therefore$ square section is economical than circular section.

NOTE: Out of the three sections rectangular section is economical

Ex:32. Two beams are simply supported over the same span and have the same flexural strength. Compare the weights of these two beams, if one of them is solid and the other is hollow circular with internal diameter half of the external diameter.



- Since both the beams are supported over the same span (l) and have the same flexural strength, therefore section modulus of both the beams must be equal.

i.e., $\frac{\pi}{32} D^3 = \frac{\pi}{32\, D_1}\{{D_1}^4 - d^4\}$

$$= \frac{\pi}{32\, D_1}\{{D_1}^4 - (0.5{D_1})^4\}$$

$$= \frac{\pi}{32} \times 0.9375{D_1}^3$$

$$\therefore D = 0.98\ D_1$$

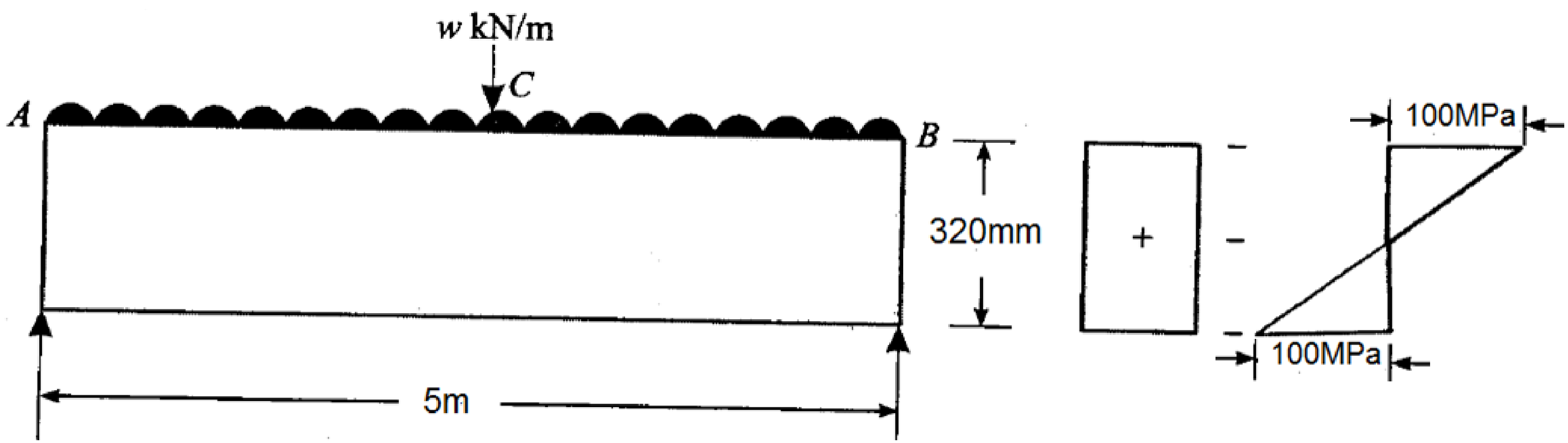
$$\frac{Weight\ of\ solid\ beam}{Weight\ of\ hollow\ beam} = \frac{Area\ of\ solid\ beam}{Area\ of\ hollow\ beam}$$

$$= \frac{\frac{\pi}{4}D^2}{\frac{\pi}{4}\{{D_1}^2 - d^2\}}$$

$$= \frac{(0.98\ D_1)^2}{\{{D_1}^2 - (0.5D_1)^2\}}$$

$$= 1.28$$

Ex:33. A rectangular beam 320 mm deep is simply supported over a span of 5 m. What uniformly distributed load the beam may carry, if the bending stress is not to exceed 100 MPa. Take I = 250 X 10^6 mm^4.



Soln.

Section modulus, $Z = I/y = 250 \times 10^6 / 160$

$= 1562500\ mm^3.$

$\therefore$ Moment of resistance, $M = f_{max} \times Z$

$= 100 \times 1562500$

$= 156.25 \times 10^6\ mm^4.$

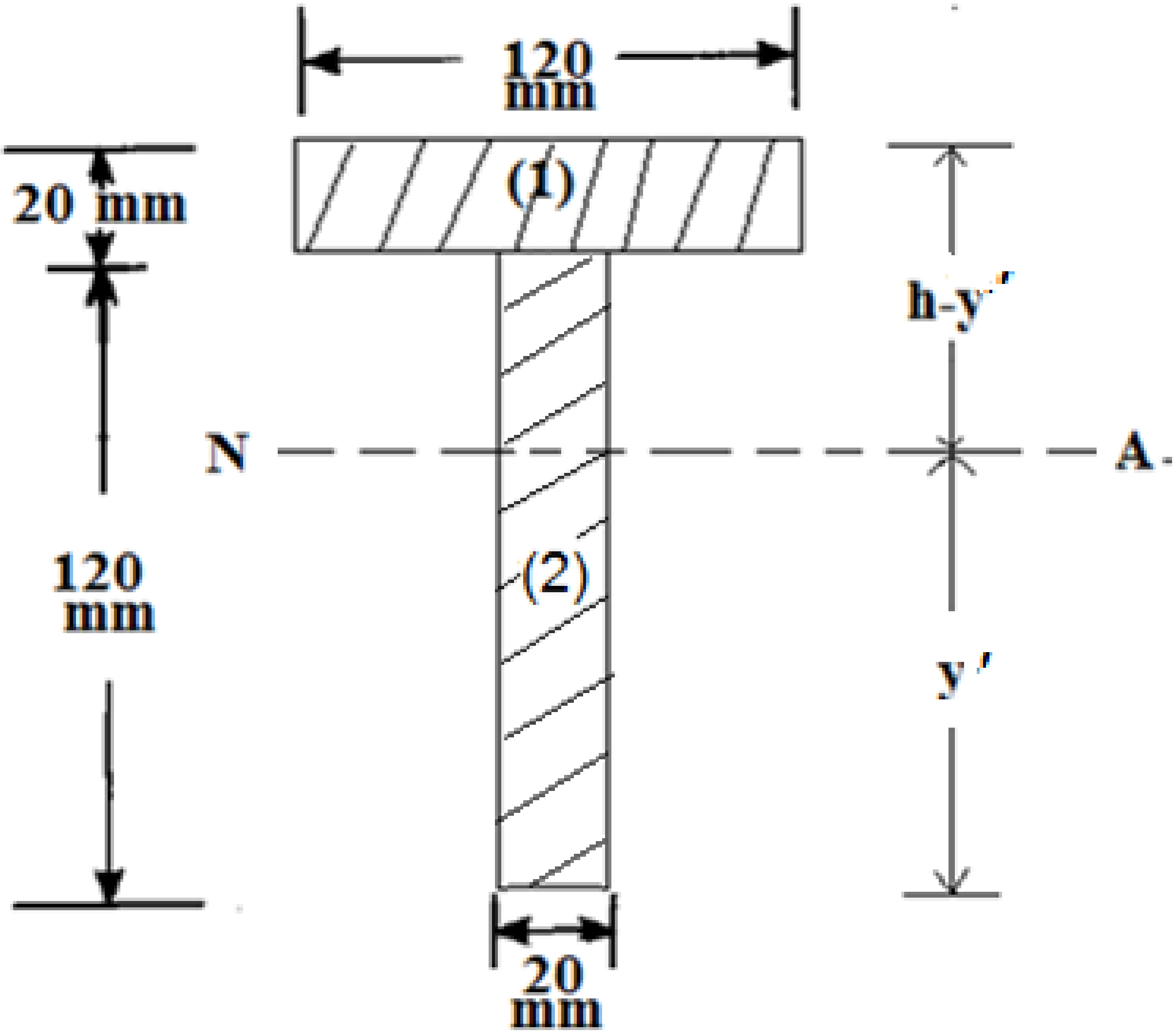
We know that maximum bending moment,

$$M = wl^2/8$$

$$\therefore 156.25 \times 10^6 = w \times \frac{5000^2}{8}$$

$\therefore$ W= 50N/mm

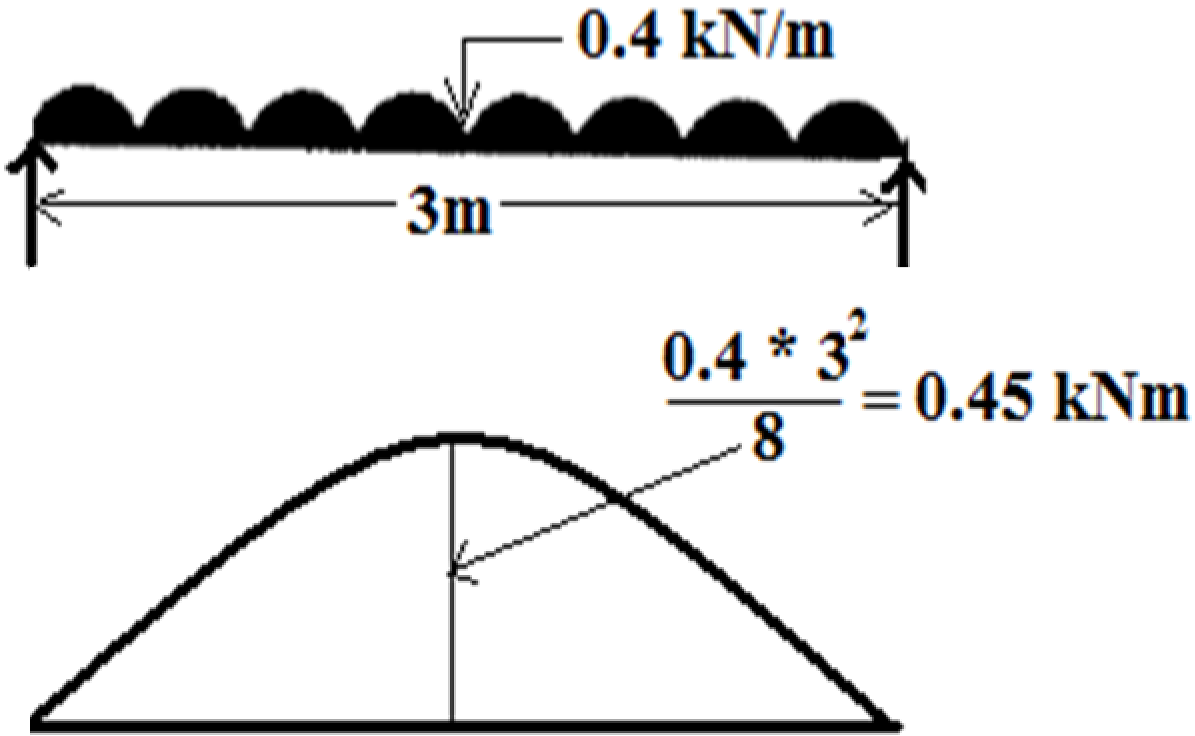
<u>Ex:34.</u> A beam section is of T-shape, formed by 2 planks of 120 mm and 20 mm. The beam is simply supported over a span of 3m and subjected to a u,d,l of 0.4kN/m. Draw the bending stress distribution diagram across the section at centre.



120 mm
20 mm
(1)
h-y′
N
A
120 mm
(2)
y′
20 mm

Soln:

$$M = 0.45 \text{ kNm}$$
$$= 0.45 \times 10^6 \text{ Nmm}$$



$$y' = \frac{a_1 y_1 + a_2 y_2}{(a_1 + a_2)}$$

$$= \frac{120 \times 20(y_1 + y_2)}{(120 \times 20) + (120 \times 20)}$$

$$= \frac{2400\big(60 + (120 + 10)\big)}{4800}$$

$$= 95\, mm$$

The neutral axis is at a depth of 95 mm from bottom.

We know, $$f = \frac{M}{I_{NA}} y$$

Where,

$$I_{NA} = (I_{XX})_1 + (I_{XX})_2$$

$$= \left(I_{X_1X_1} + a_1 {h_1}^2\right) + \left(I_{X_2X_2} + a_2 {h_2}^2\right)$$

$$= \frac{1}{12} \times 20 \times 120^3 + 120 \times 20 \times 35^2 + \frac{1}{12} \times 120 \times 20^3 + 120 \times 20 \times 35^2$$

$$= 8.84 \times 10^6\, mm^4$$

$$f_{c\,max} = \frac{M}{I} y_{c\,max}$$

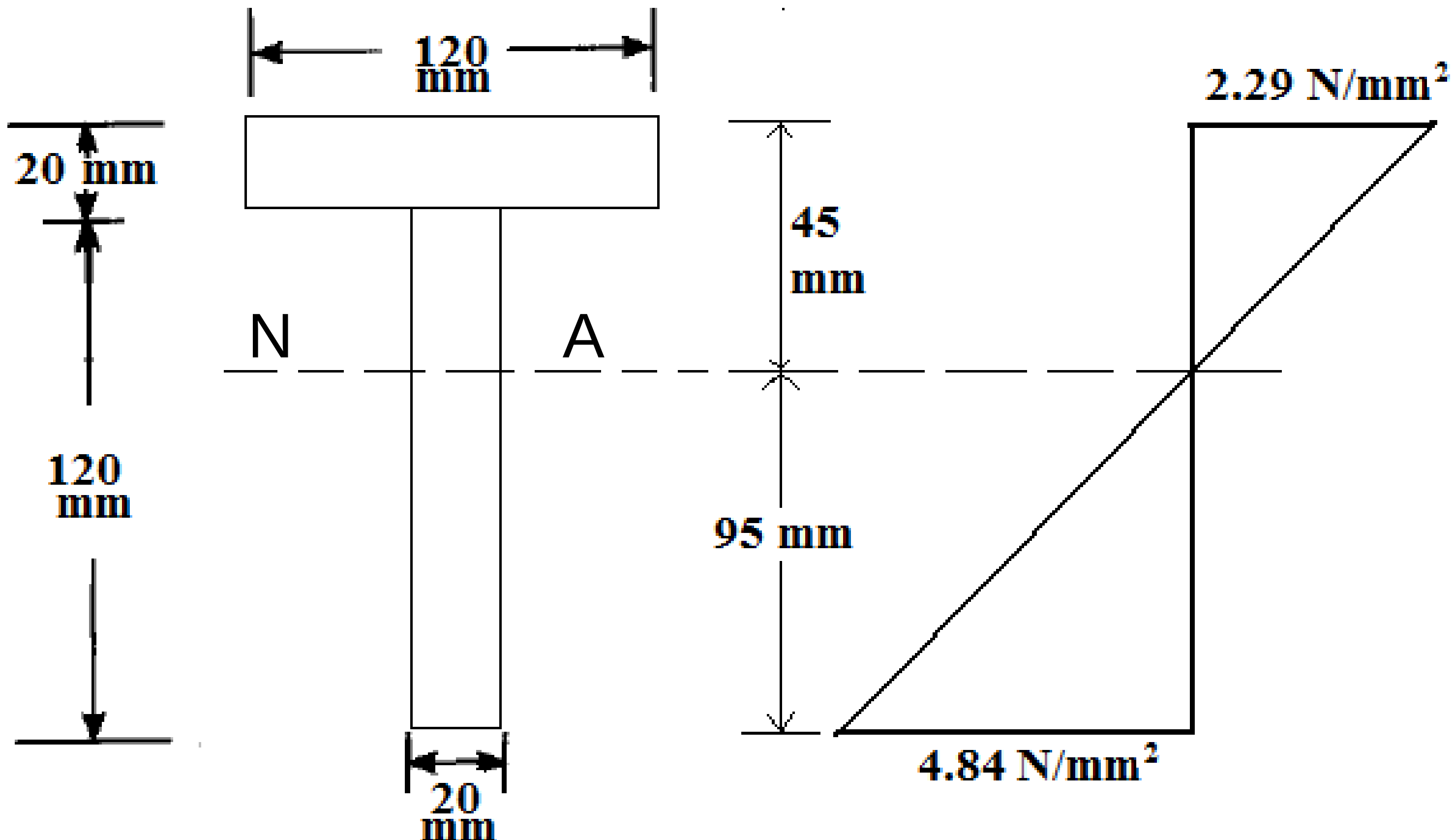
$$= \frac{0.45 \times 10^6 \times 45}{8.84 \times 10^6}$$

$$= 2.29\ N/mm^2$$

$$f_{t\,max} = \frac{M}{I}\ y_{tmax}$$

$$f_{t\,max} = \frac{0.45 \times 10^6 \times 95}{8.84 \times 10^6}$$

$$= 4.84\ N/mm^2$$

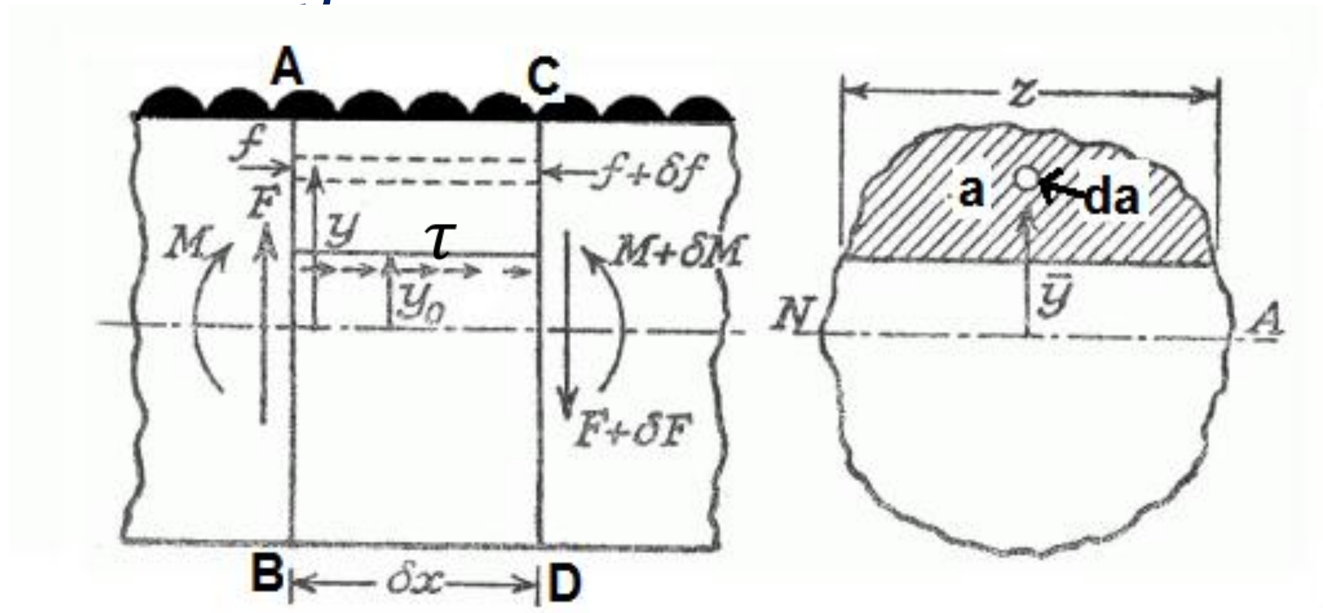


Bending stress distribution

SHEAR STRESS DISTRIBUTION

Shearing stresses at a section in a loaded beam:

Consider a small portion *ABDC of length dx of a beam loaded with uniformly distributed* load as shown in Fig.



when a beam is loaded with a uniformly distributed load, the shear force and bending moment vary at every point along the length of the beam.

Let, M = *Bending moment at AB,*

$M + dM$ = *Bending moment at CD,*

F = *Shear force at AB,*

$F + dF$ = *Shear force at CD, and*

I = second moment of area of the section about its neutral axis.

Now consider an elementary strip at a distance *y from the neutral axis.*

Let, f = Intensity of bending stress across *AB at distance y from the neutral axis*

f + df = Intensity of bending stress across CD

and

da = Cross-sectional area of the strip.

We know $$\frac{M}{I} = \frac{f}{y}$$

Therefore intensity of bending stress on the left side of the strip, $$f = \frac{M}{I} \times y$$

Similarly, intensity of bending stress on the right side of the strip,

$$f + df = \frac{M + dM}{I} \times y$$

We know that the force acting on the strip across AB

$$= \text{Stress} \times \text{Area} = f \times da = \frac{M}{I} \times y \times da$$

Similarly, force acting on the strip across CD

$$= (f + df) \times da$$

$$= \frac{M + dM}{I} \times y \times da$$

$\therefore$ Net unbalanced force on the strip

$$= \frac{M + dM}{I} \times y \times da - \frac{M}{I} \times y \times da = \frac{dM}{I} \times y \times da$$

The total unbalanced force (F) at any layer at a distance y_0 from the neutral axis can be found out by integrating the above equation between y_0 to y_1.

i.e., F $= \int_{y_0}^{y_1} \frac{dM}{I} da \times y\, dy = \frac{dM}{I} a\bar{y}$

Where, $a\bar{y}$ = *first moment of area of the section above the layer taken about the neutral axis.*

We know that the intensity of the shear stress,

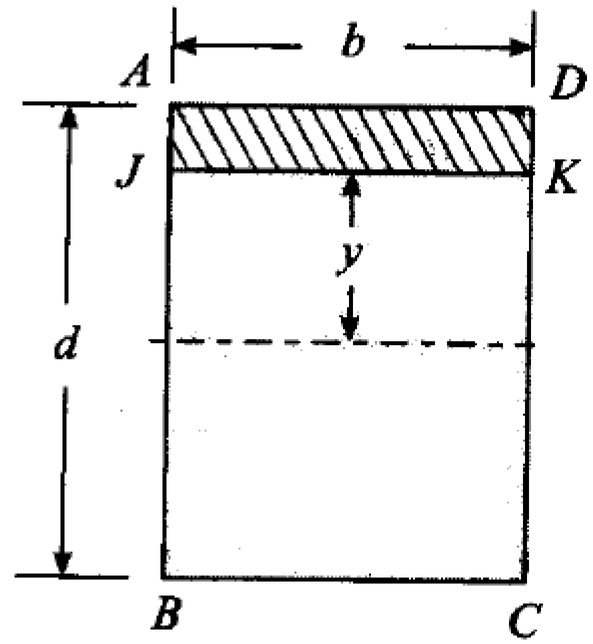
$$\tau = \frac{Total\ force}{Area} = \frac{\frac{dM}{I}\ a\overline{y}}{dx.b}$$

$$= \frac{dM}{dx} \times \frac{a\overline{y}}{Ib}$$ (where b is the width of layer)

$$= F \times \frac{a\overline{y}}{Ib}\ \left(Substituting\ \frac{dM}{dx} = F = Shear\ force\right)$$

Distribution of Shearing stress over a Rectangular Section

Consider a beam of rectangular section ABCD of width ‘b’ and depth ‘d’ as shown in Fig.



Rectangular section

We know that the shear stress on a layer *JK of beam, at a distance y from the neutral axis,*

$$\tau = F \times \frac{a\bar{y}}{Ib}$$

where *F = Shear force at the section,*

$a\bar{y}$ = first moment of the shaded area about NA

I = second moment of area of the whole section about its neutral axis,

and

b = Width of the section.

We know that area of the shaded portion *AJKD*,

$$a = b\left(\frac{d}{2} - y\right)$$

$$\bar{y} = y + \frac{1}{2}\left(\frac{d}{2} - y\right) = y + \frac{d}{4} - \frac{y}{2}$$

$$= \frac{y}{2} + \frac{d}{4} = \frac{1}{2}\left(y + \frac{d}{2}\right)$$

Substituting the above values,

$$\tau = F \times \frac{a\bar{y}}{Ib}$$

$$= F \times \frac{b\left(\frac{d}{2} - y\right) \times \frac{1}{2}\left(y + \frac{d}{2}\right)}{Ib}$$

$$= \frac{F}{2I}\left(\frac{d^2}{4} - y^2\right)$$

- From the above equation, that shear stress increase as *y decreases.*

- *At a point, where y = d/2, shear stress= 0*
- *We also see that the variation of shear stress with respect to y is a parabola.*

At neutral axis i.e., at y = 0, the value of shear stress is maximum.

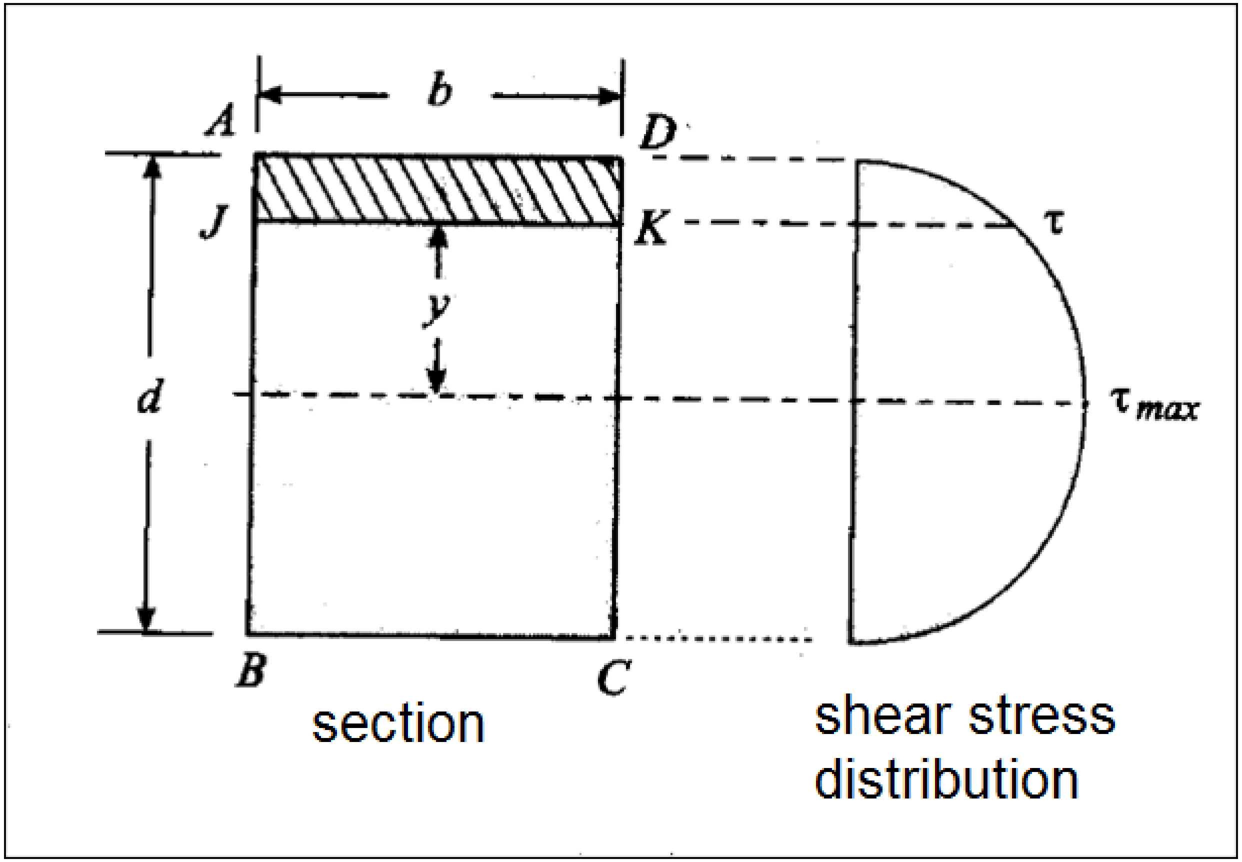
Thus substituting *y = 0 and I = b d* ^{3} */ 12* in the above equation,

$$\tau_{max} = \frac{F}{2 \times \frac{bd^3}{12}}\left(\frac{d^2}{4}\right) = \frac{3F}{2bd} = 1.5\tau_{av}$$

where,

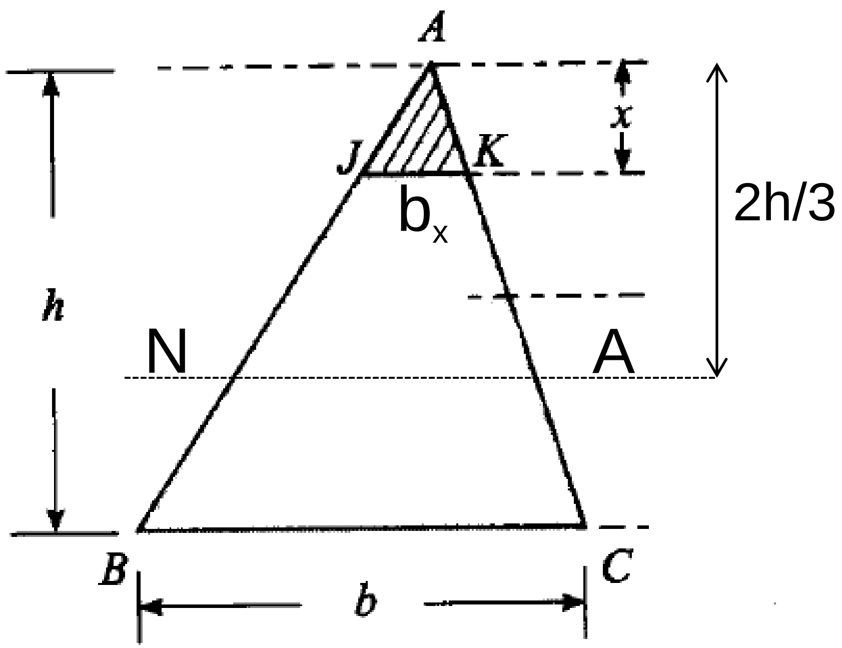
$$(\tau_{av} = \frac{F}{Area} = \frac{F}{bd})$$

The shear stress distribution diagram is shown in Fig.



Distribution of Shearing stress over a Triangular Section

Consider a beam of triangular cross section ABC of base ‘b’ and height ‘h’ as shown in Fig.



Triangular section

We know that the shear stress on a layer JK at a distance y from the neutral axis,

$$\tau = F \times \frac{a\bar{y}}{I b_x}$$

Where, F = Shear force at the section,

$a\bar{y}$ = first moment of the shaded area about the neutral axis and

I = second Moment of area of the triangular section about its neutral axis.

We know that width of the strip *JK*,

$$b_x = \frac{bx}{h}$$

$\therefore$ Area of the shaded portion AJK,

$$a = \frac{1}{2} JK \times x = \frac{1}{2}\left(\frac{bx}{h} \times x\right) = \frac{bx^2}{2h}$$

Centroid of the shaded area from the neutral axis,

$$\overline{y} = \frac{2h}{3} - \frac{2x}{3} = \frac{2}{3}(h - x)$$

Substituting the values of b_x, a and y

$$\tau = F \times \frac{\left(\frac{bx^2}{2h}\right) \times \frac{2}{3}\,(h - x)}{I \times \frac{bx}{h}}$$

$$= \frac{F}{3I} \times x(h - x)$$

$$= \frac{F}{3I} \times (hx - x^2)$$

Thus we see that the variation of shear stress is a parabola.

We also see that as a point where x = 0 or x = h, shear stress= 0.

At neutral axis, where x = 2h/3,

$$\tau = \frac{F}{3I}\left[h \times \frac{2h}{3} - \left(\frac{2h}{3}\right)^2\right] = \frac{F}{3I} \times \frac{2h^2}{9} = \frac{2Fh^2}{27I}$$

$$= \frac{2Fh^2}{27 \times \frac{bh^3}{36}} = \frac{8F}{3bh} \qquad \left(I = \frac{bh^3}{36}\right)$$

$$\left(Area = \frac{bh}{2}\right)$$

$$= \frac{4}{3} \times \frac{F}{Area} = 1.33\ \tau_{av}$$

Now for maximum intensity, differentiating the shear stress *and equating to zero, we get*

$$\frac{d\tau}{dx}\left[\frac{F}{3I} \times (hx - x^2)\right] = 0$$

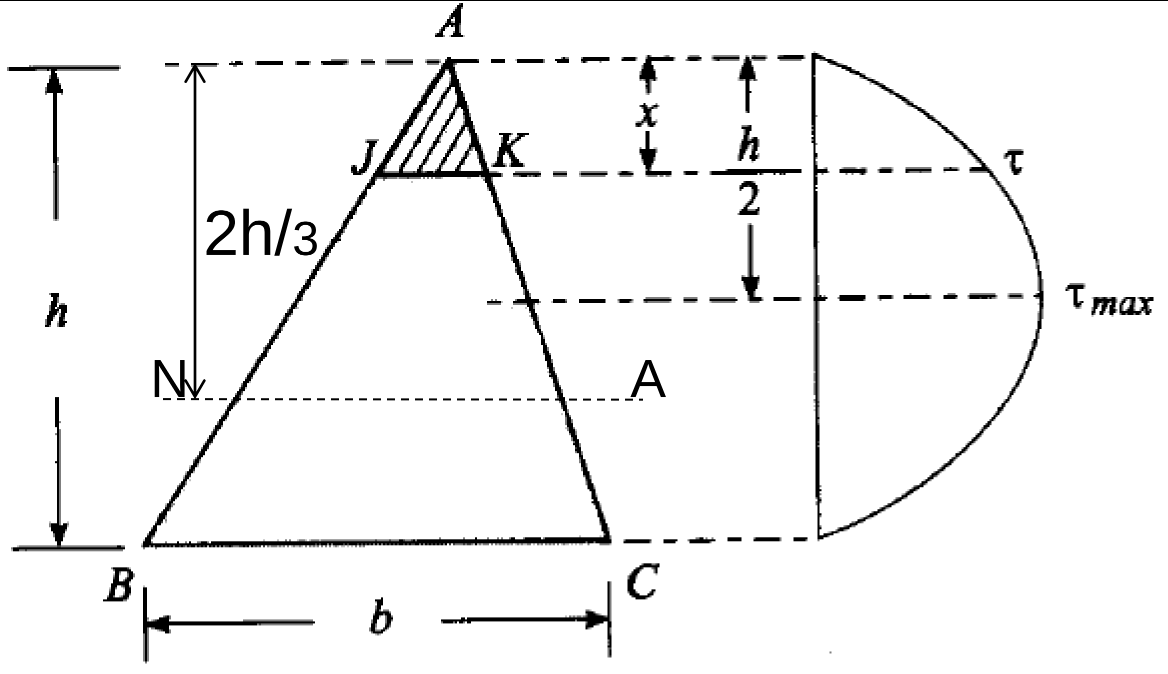
$$h - 2x = 0 \quad (or) \quad x = \frac{h}{2}$$

Now substituting the value of x in the equation of shear stress,

$$\tau_{max} = \frac{F}{3I}\left[h \times \frac{h}{2} - \left(\frac{h}{2}\right)^2\right] = \frac{Fh^2}{12I} \qquad \left(I = \frac{bh^3}{36}\right)$$

$$= \frac{Fh^2}{12 \times \frac{bh^3}{36}} = \frac{3F}{bh} = \frac{3}{2} \times \frac{F}{Area} = 1.5\ \tau_{av}$$

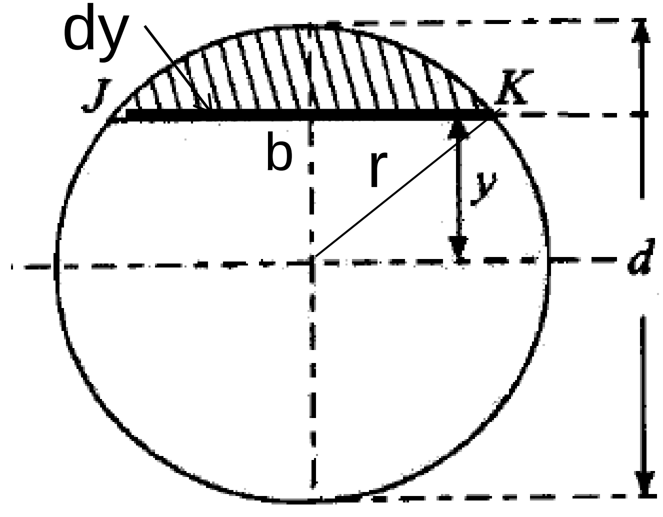
The shear stress distribution diagram is shown in Fig.



Triangular section.

Distribution of Shearing stress over a Circular Section

Consider a circular section of diameter 'd' as shown in Fig.



Circular section

We know that the shear stress on a layer JK at a distance y from the neutral axis,

$$\tau = F \times \frac{a\bar{y}}{Ib}$$

Where, F = Shear force at the section,

$a\bar{y}$ = Moment of the shaded area about the neutral axis,

r = Radius of the circular section,

I = second moment of area of the circular section about the neutral axis,

b = Width of the strip JK.

We know that in a circular section,

width of the strip *JK*,

$$b = 2\sqrt{r^2 - y^2}$$

and area of the strip,

$$da = 2\sqrt{r^2 - y^2}.dy$$

$\therefore$ Moment of this area about the neutral axis

$$= 2y\sqrt{r^2 - y^2}.dy$$

Now moment of the whole shaded area about the neutral axis may be found out by integrating above equation between the limits *y and r, i.e.,*

$$a\bar{y} = \int_y^r 2y\sqrt{r^2 - y^2}.\,dy$$

$$= \int_y^r b.\,y.\,dy \qquad (b = 2\sqrt{r^2 - y^2})$$

We know that width of the strip *JK*,

$$b = 2\sqrt{r^2 - y^2}$$

$$b^2 = 4(r^2 - y^2)$$

Differentiating both sides of the above equation,

$$2b \cdot db = 4\,(-\,2y)\,dy = -\,8y \cdot dy$$

$$y \cdot dy = -1/4.b.db$$

Substituting the value of '*ydy*',

$$a\overline{y} = \int_{y}^{r} b\left(-\frac{1}{4}b.db\right) = -\frac{1}{4}\int_{y}^{r} b^{2}.db$$

We know that when y = y, width b = b and when y = r, width b = 0. Therefore, the limits of integration may be changed as "b to zero" in the above equation.

$$a\bar{y} = -\frac{1}{4}\int_b^0 b^2 \, . \, db$$

$$= \frac{1}{4}\int_0^b b^2 \, . \, db \quad \text{(Eliminating – ve sign)}$$

$$= -\frac{1}{4}\left[\frac{b^3}{3}\right]_0^b$$

$$= \frac{b^3}{12}$$

Now substituting this value of $a\bar{y}$ in the original formula for the shear stress, i.e.,

$$\tau = F \times \frac{a\bar{y}}{Ib} = F \times \frac{\frac{b^3}{12}}{Ib} = F \times \frac{b^2}{12I}$$

$$= F \times \left[\frac{\left(2\sqrt{r^2 - y^2}\right)^2}{12I} \right] \qquad \left(b = 2\sqrt{r^2 - y^2}\right)$$

$$= F \times \frac{r^2 - y^2}{3I}$$

Thus we again see that shear stress increases as *y decreases.*

At a point, where y = r, shear stress= 0 and the shear stress is maximum where y is zero(at neutral axis).

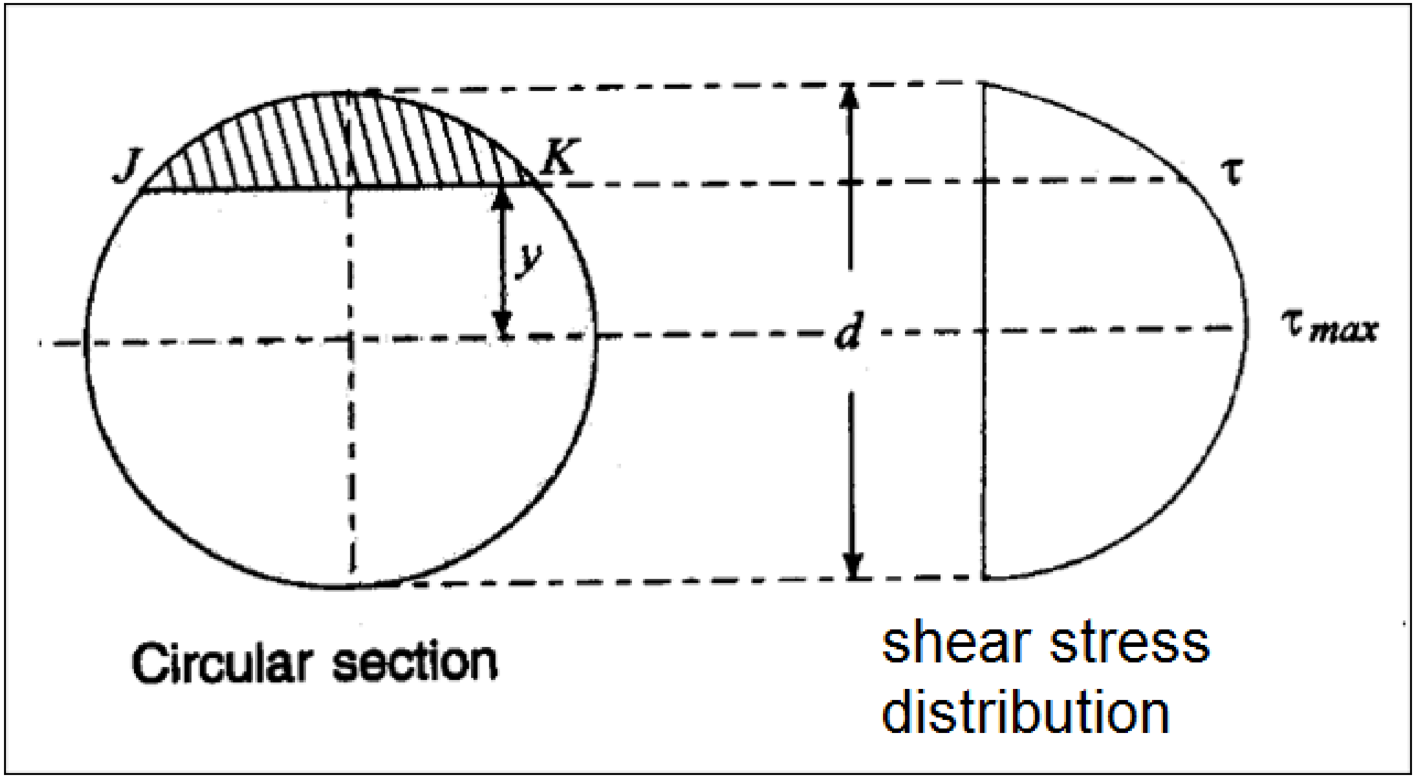
We also see that the variation of shear stress is a parabolic curve.

To find the value of maximum shear stress substitute y=0 and $I = \frac{\pi}{64} \times d^4$ in the above equation, we get,

$$\tau_{max} = F \times \frac{r^2}{3 \times I} = F \times \frac{\left(\frac{d}{2}\right)^2}{3 \times \frac{\pi}{64} \times d^4}$$

$$= \frac{4F}{3 \times \frac{\pi}{4} \times d^2} = 1.33\ \tau_{av}$$

The shear stress distribution diagram is shown in Fig.



Ex:35. *A wooden beam 100 mm wide x 250 mm deep* and 3 m long is simply supported *at its ends. It carries a u.d.l of 40 kN/m over the entire length. Determine the maximum shear stress and sketch the variation of shear stress along the depth of the beam.*

Sol:

uniformly distributed load *(w) = 40 kN/m*

= 40 N/mm.

We know that maximum shear force,

$$F = \frac{wl}{2} = \frac{40 \times (3 \times 10^3)}{2}\ N$$

$$= 60 \times 10^3\ N$$

and area of beam section,

$$A = b \cdot d = 100 \times 250 = 25\,000\ mm^2$$

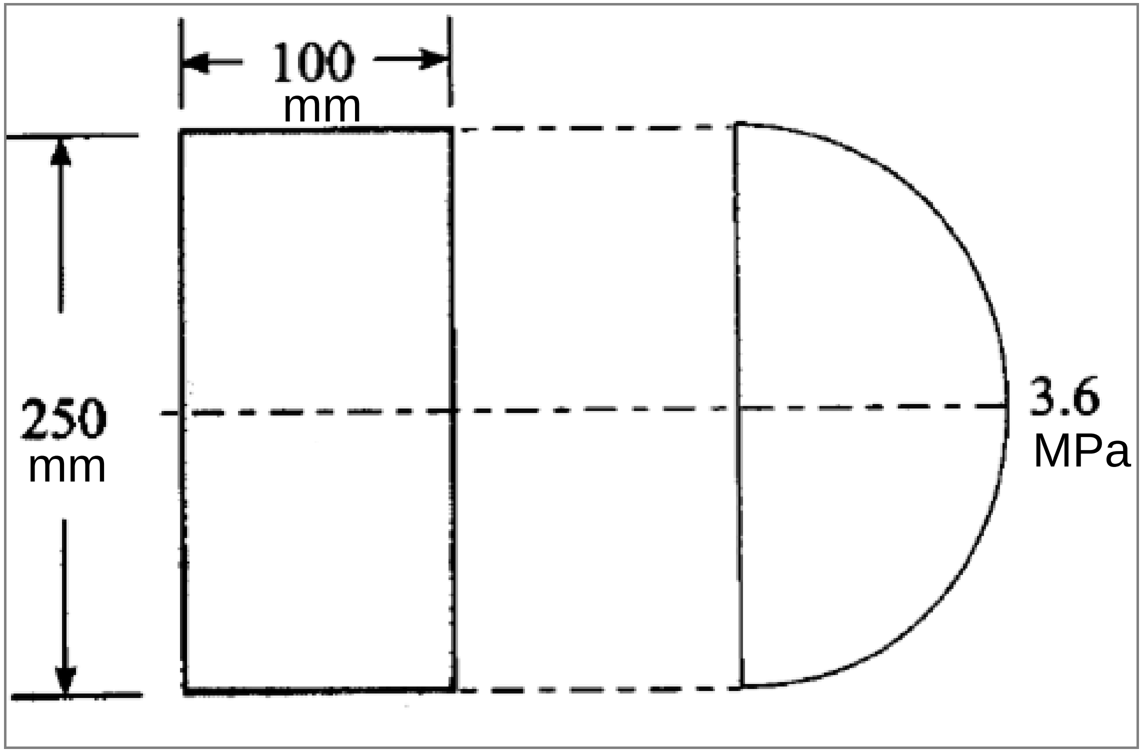
∴ Average shear stress across the section

$$\tau_{av} = \frac{F}{A} = \frac{60 \times 10^3}{25000} = 2.4N/mm^2 = 2.4\ MPa$$

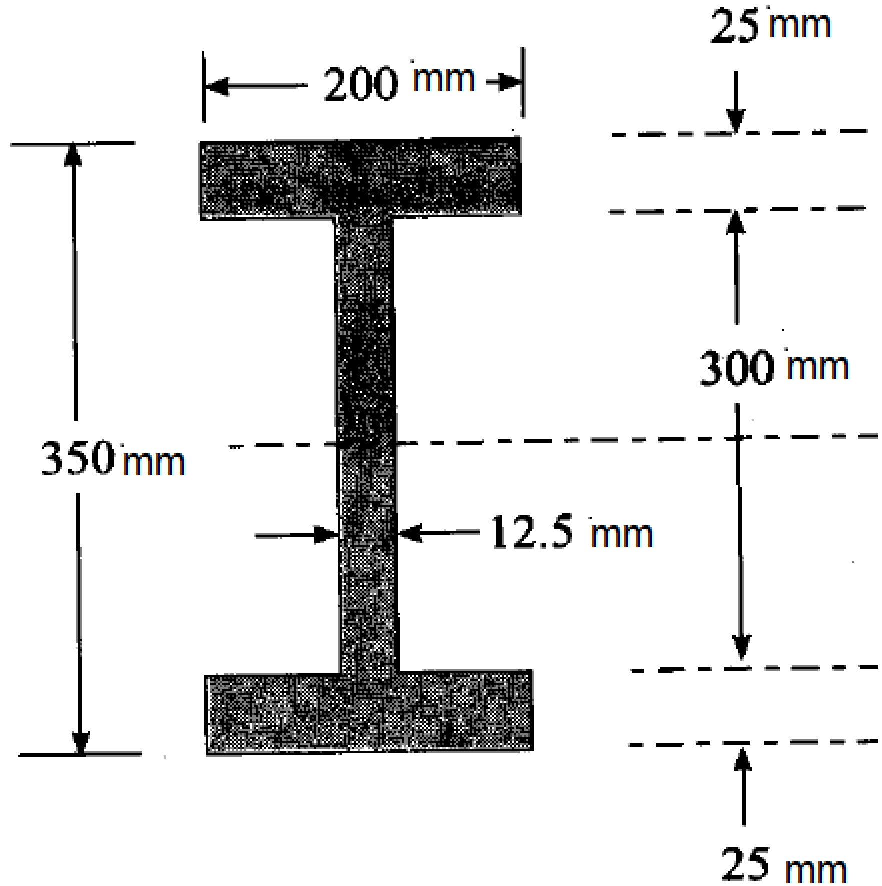
and maximum shear stress,

$$\tau_{max} = 1.5 \times \tau = 1.5 \times 2.4 = 3.6\ MPa$$

The diagram showing the variation of shear along the depth of the beam is shown in Fig.



Ex: 36. *An I-section as shown in Fig is subjected to a shearing force of 200 kN. Sketch the shear stress distribution across the section.*



We know that the second moment of area of the I-section about it horizontal centroidal axis (*x*-*x*),

$$I_{XX} = \frac{200 \times (350)^3}{12} - \frac{187.5 \times (300)^3}{12}$$

$$= 292.7 \times 10^6\ mm^4$$

We also know that shear stress at the upper edge of the upper flange is zero.

shear stress at the lower edge of the upper flange (junction of the top flange and web)

$$= \frac{200 \times 10^3 \times (200 \times 25 \times 162.5)}{292.7 \times 10^6 \times 200}$$

$=2.79$ N/mm^2

.

shear stress at the upper edge of the web

(junction of the top flange and web)

$$= \frac{200 \times 10^3 \times (200 \times 25 \times 162.5)}{292.7 \times 10^6 \times 12.5}$$

$= 44.41\ \text{N}/mm^2$

The shear stress at the junction suddenly increases from 2.79 MPa to 44.41 MPa.

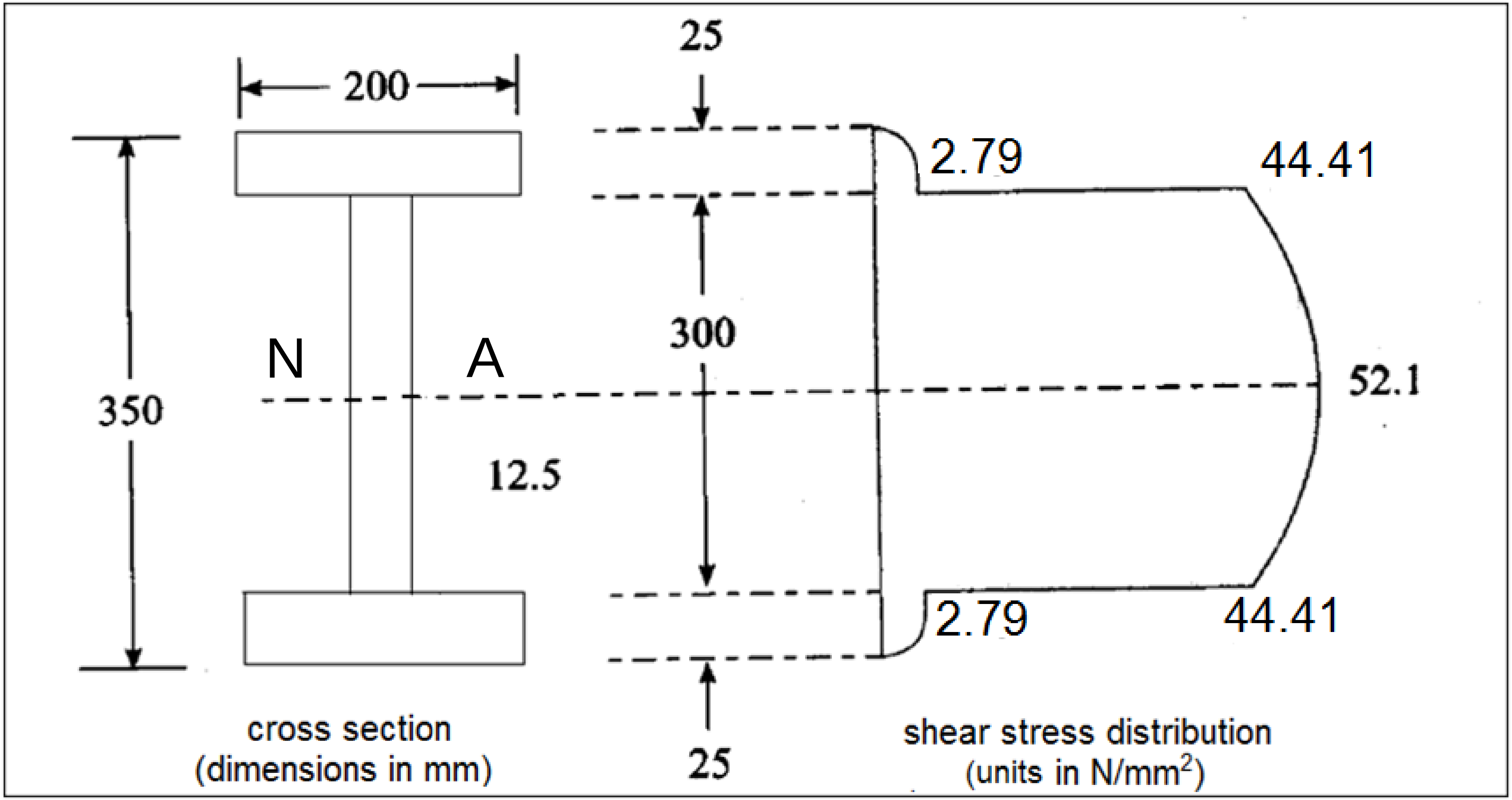
shear stress at the neutral axis

$$= \frac{200 \times 10^3 \times (200 \times 25 \times 162.5 + 150 \times 12.5 \times 75)}{292.7 \times 10^6 \times 12.5}$$

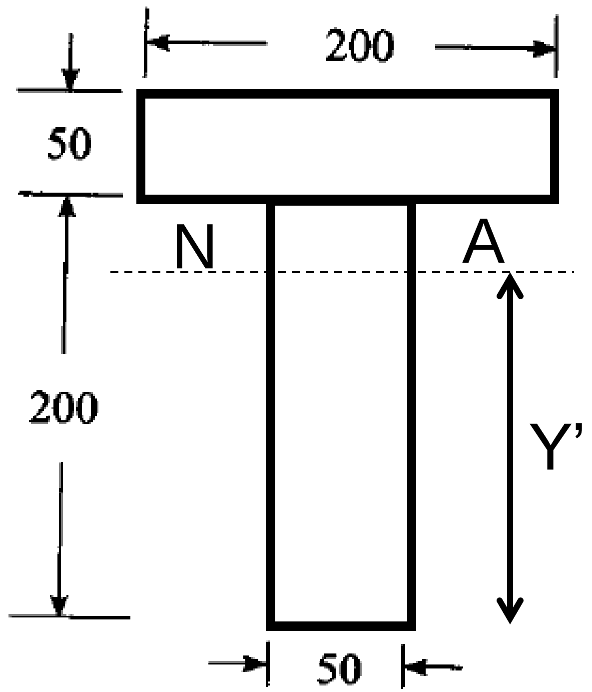
$$= 52.1\ N/mm^2$$

Due to symmetry of the section, the shear stress distribution diagram is also symmetrical about the neutral axis

The shear stress distribution diagram across the section is shown in Fig.



Ex.37. A T-shaped cross-section of a beam shown in Fig. is subjected to a vertical shear force of 100 kN. Calculate the shear stress at important points and draw shear stress distribution diagram Moment



Dimensions are in mm

Position of horizontal centroidal axis from bottom of web,

$$y' = \frac{(200 \times 50 \times 225) + (200 \times 50 \times 100)}{(200 \times 50) + (200 \times 50)}$$

=162.5 mm

$$I_{NA} = \frac{1}{12} \times 200 \times 50^3 + 200 \times 50 \times 62.5^2$$

$$+ \frac{1}{12} \times 50 \times 200^3 + 200 \times 50 \times 62.5^2$$

$$= 113.54 \times 10^6\ mm^4$$

Shear stress at top layer of flange = 0

Shear stress at bottom layer of the flange

$$= \frac{Fa\bar{y}}{Ib}$$

$$= \frac{100 \times 10^3 \times 200 \times 50 \times 62.5}{113.54 \times 10^6 \times 200}$$

$$= 2.75 \text{ N/mm}^2$$

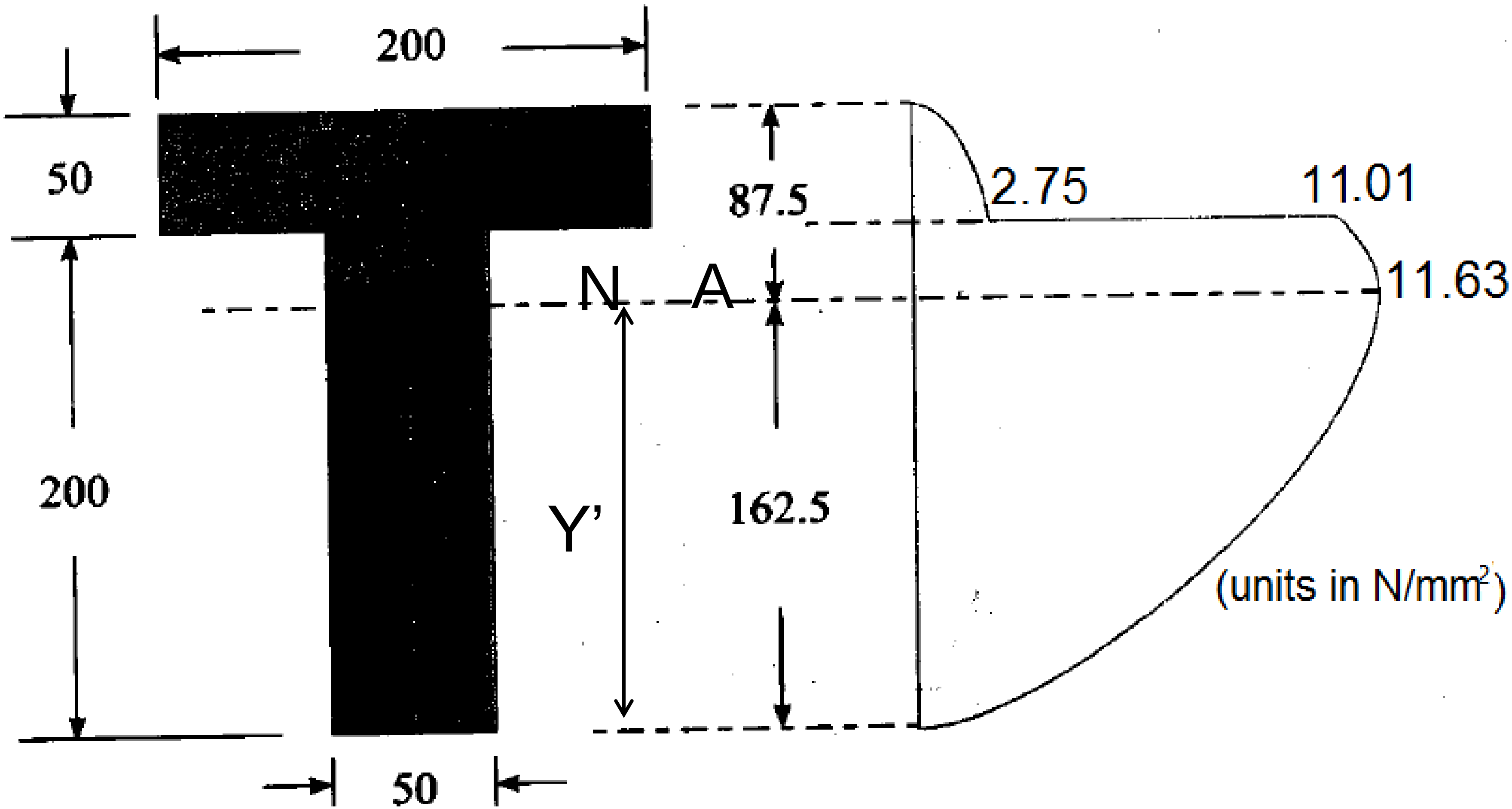
Shear stress at top most layer of web

$$= \frac{100 \times 10^3 \times 200 \times 50 \times 62.5}{113.54 \times 10^6 \times 50} = 11.01 \text{ N/mm}^2$$

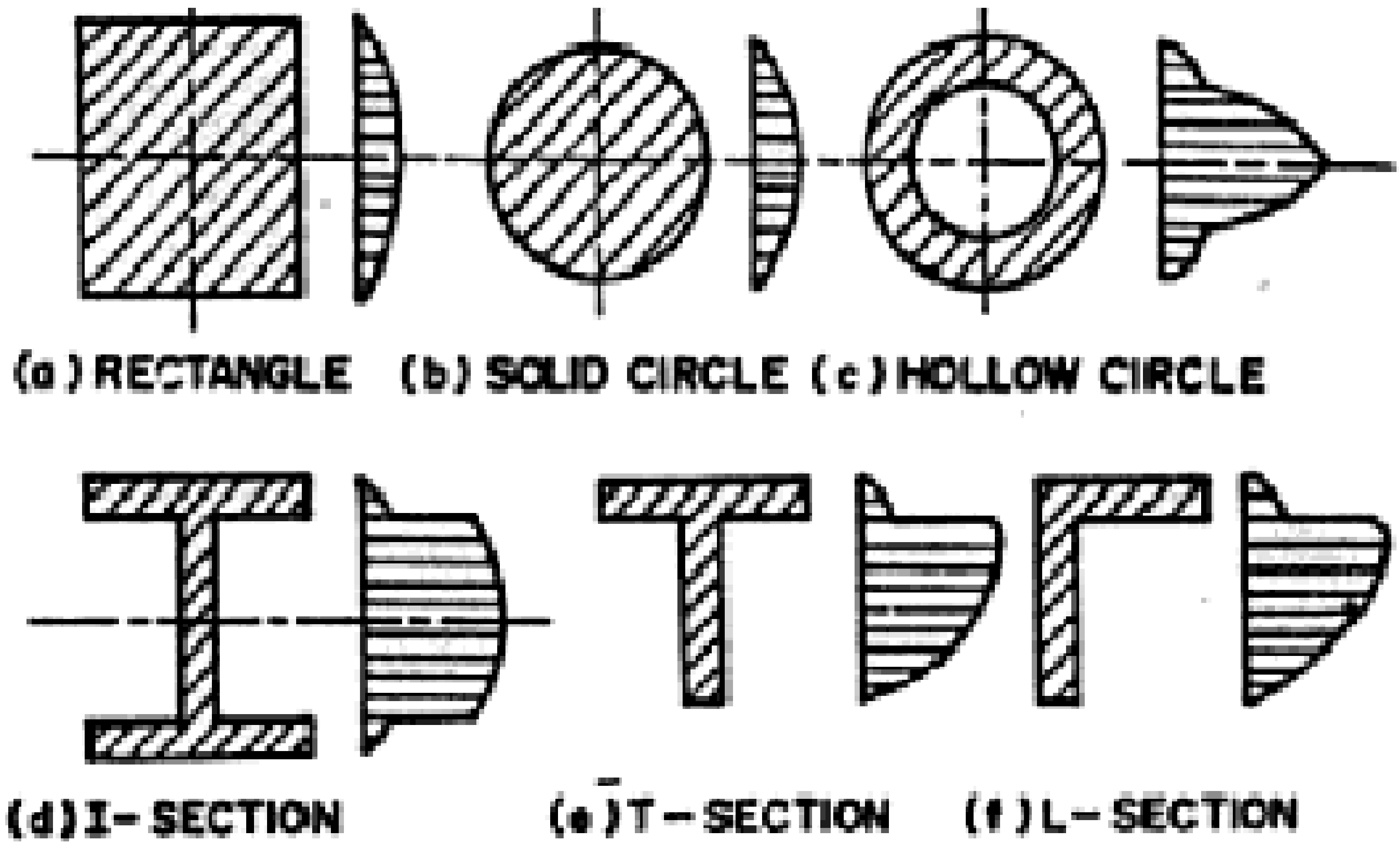
Shear stress at the neutral axis

$$= \frac{100 \times 10^3 \times 162.5 \times 50 \times \frac{162.5}{2}}{113.54 \times 10^6 \times 50} = 11.63 \text{ N/mm}^2$$

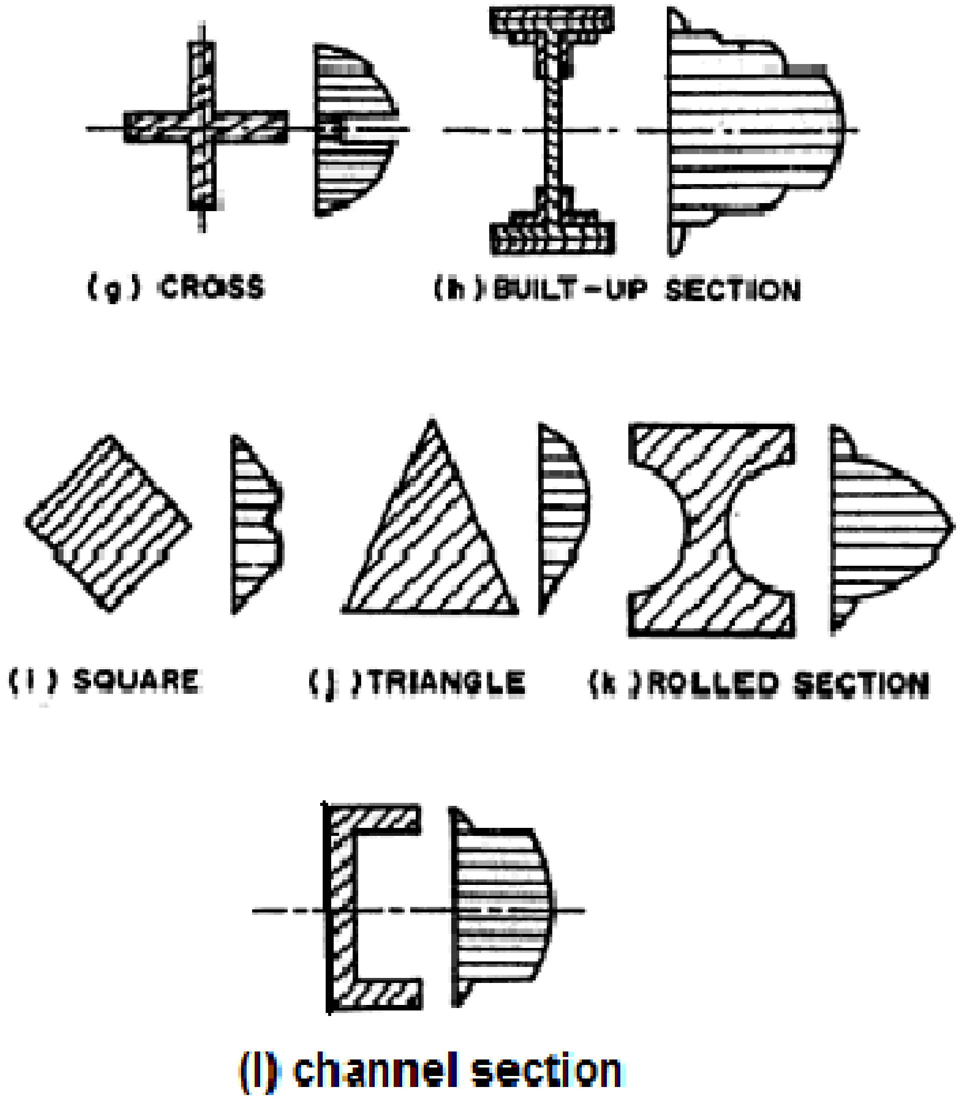
Shear stress at the bottom most layer of the web = 0



Section (dimensions in mm) shear stress distribution



Shear stress distributions over some common sections



Shear stress distributions over some common sections

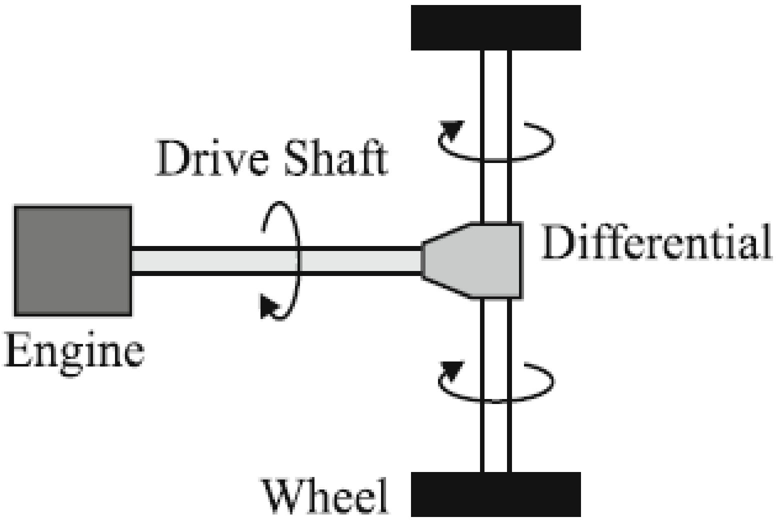
UNIT 3 TORSION

SHAFT

When you ride a bicycle, you transfer power from your leg to the pedals, and through shaft and chain to the rear wheel.

In a car, power is transferred from the engine to the wheel, requiring many shafts that form the drive train.

Shaft…



schematic of the assembly of torsion members to propel automobiles

(a)

(b)

Shaft…

A shaft also transfer torque to the blades of a helicopter.

Fig.6.2

Shaft…

“ any structural member that transmits torque from one plane to another is called a shaft”

A member is said to be in torsion when it is subjected to twisting moment about its axis.

Torque

Twisting moments or torques are forces acting through distance so as to promote rotation.

Torque

Example:

Using a wrench to tighten a nut in a bolt.

If the bolt, wrench and force are all perpendicular to one another, the twisting moment is the force (F) times the length (L) of the wrench.

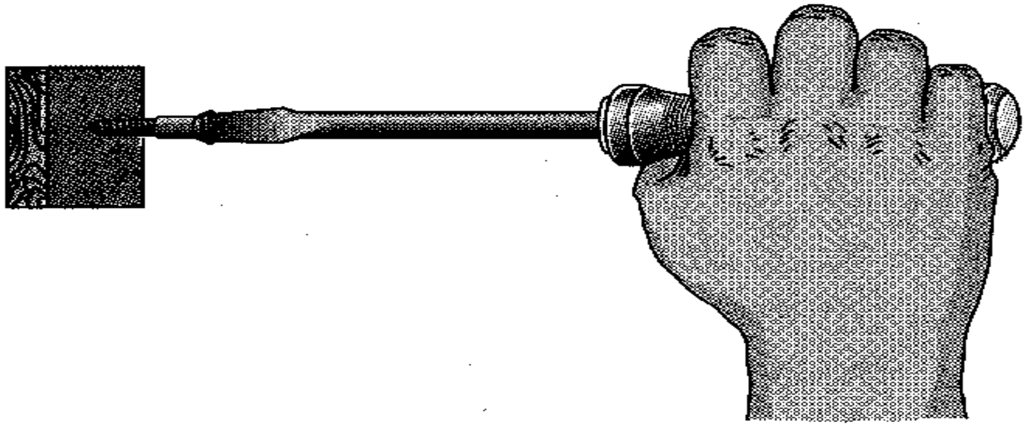
$$\text{Simple torque , } T = F \times L$$

Torsion

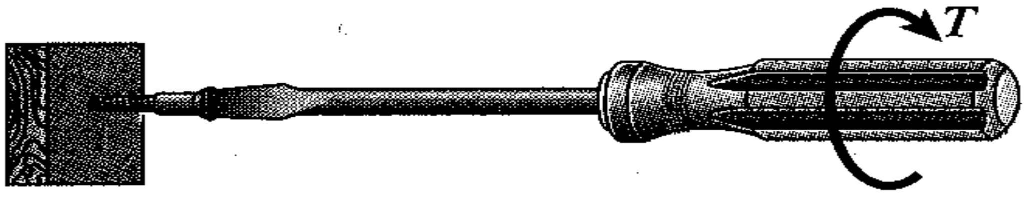
Torsion is the twisting of a straight bar when it is loaded by twisting moments or torques that tend to produce rotation about the longitudinal axis of the bar.

For instance, when we turn a screw driver to produce torsion, our hand applies torque ‘T’ to the handle and twists the shank of the screw driver.

- Torsion



(a)



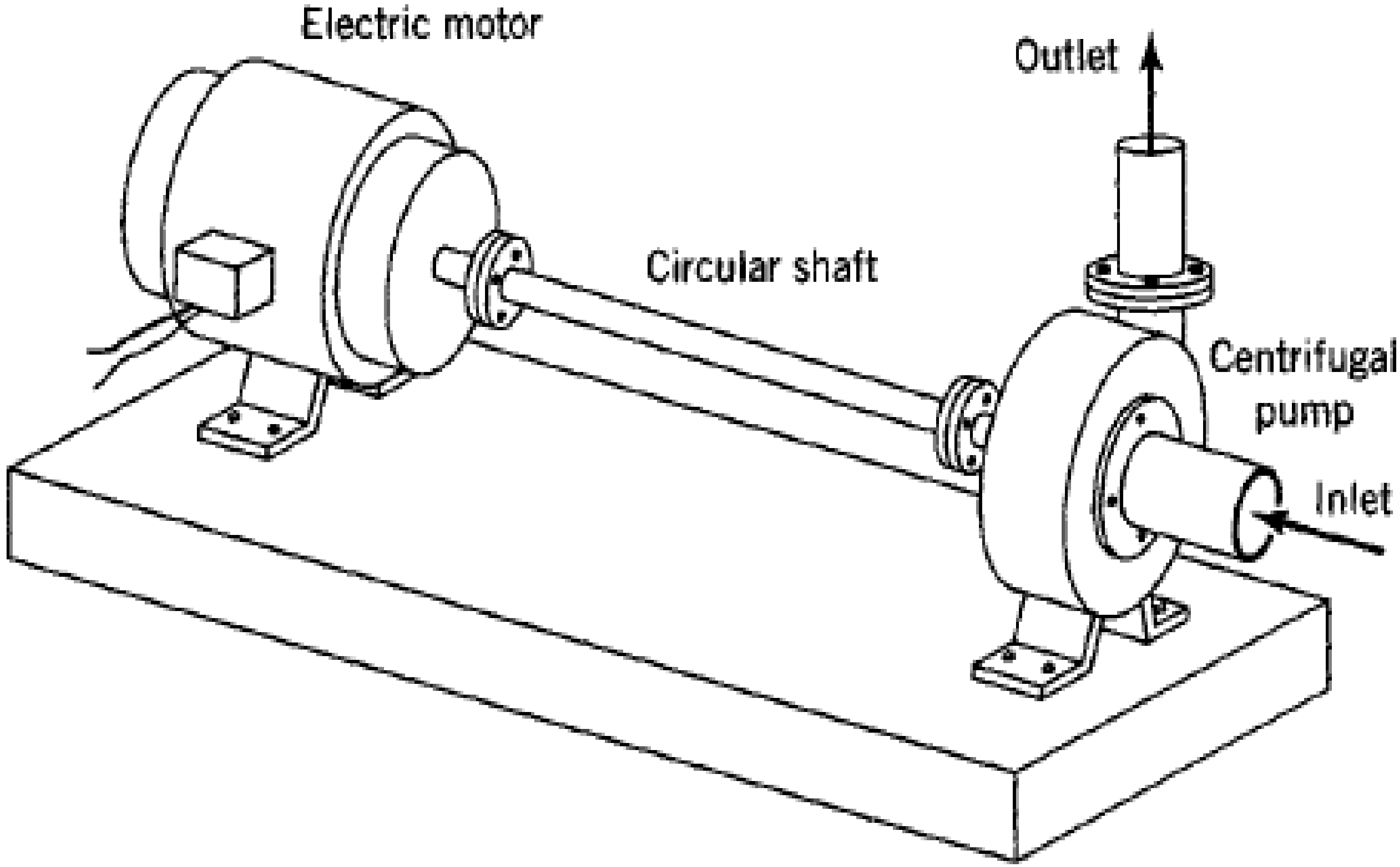
(b)

Torsion

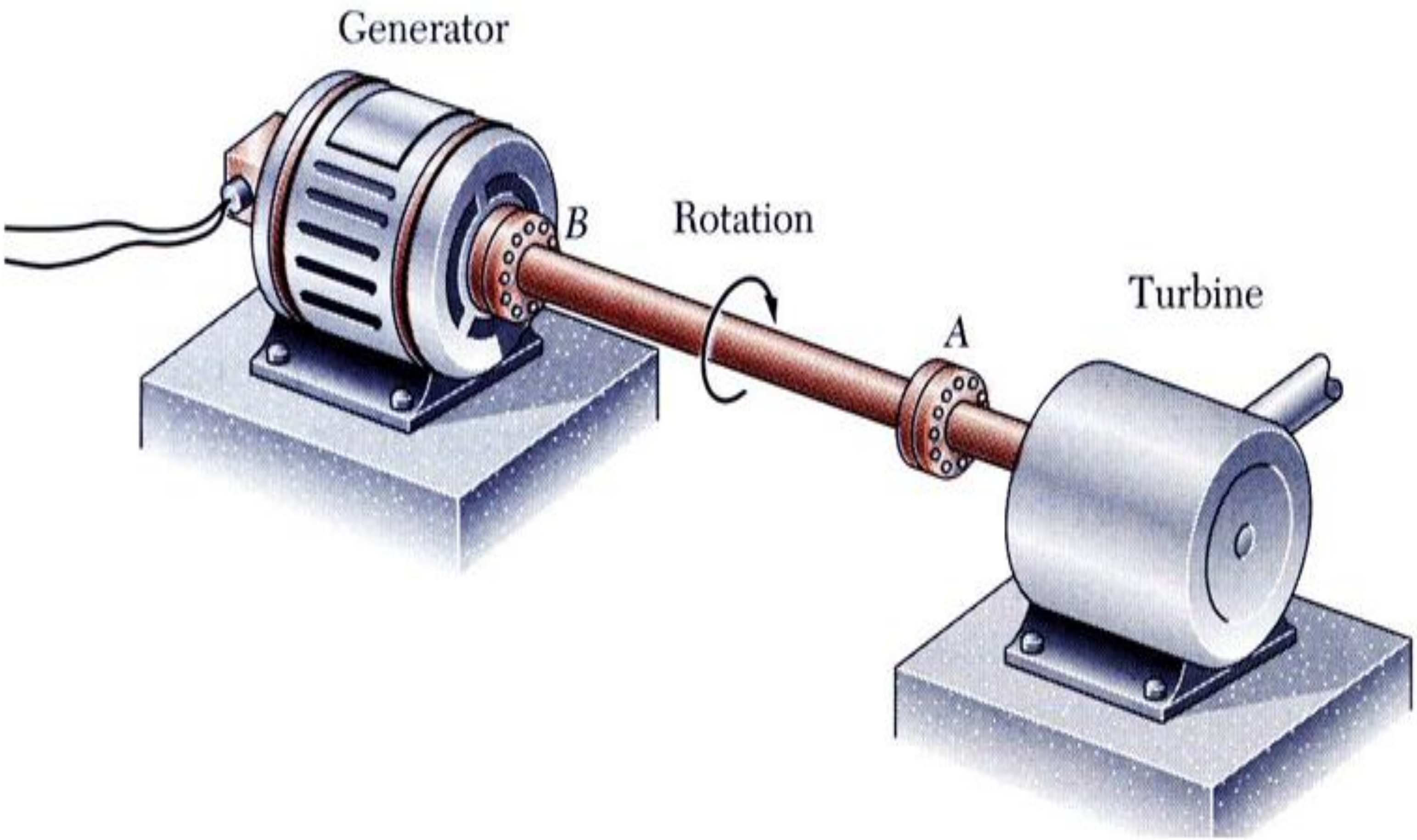
In engineering problems many members are subjected to torsion.

Shafts transmitting power from engine to the rear axle of automobile, from a motor to machine tool and from a turbine to electric motors are the common examples of members in torsion.

A common method of transmitting power is by means of ***torque*** in a rotating shaft.

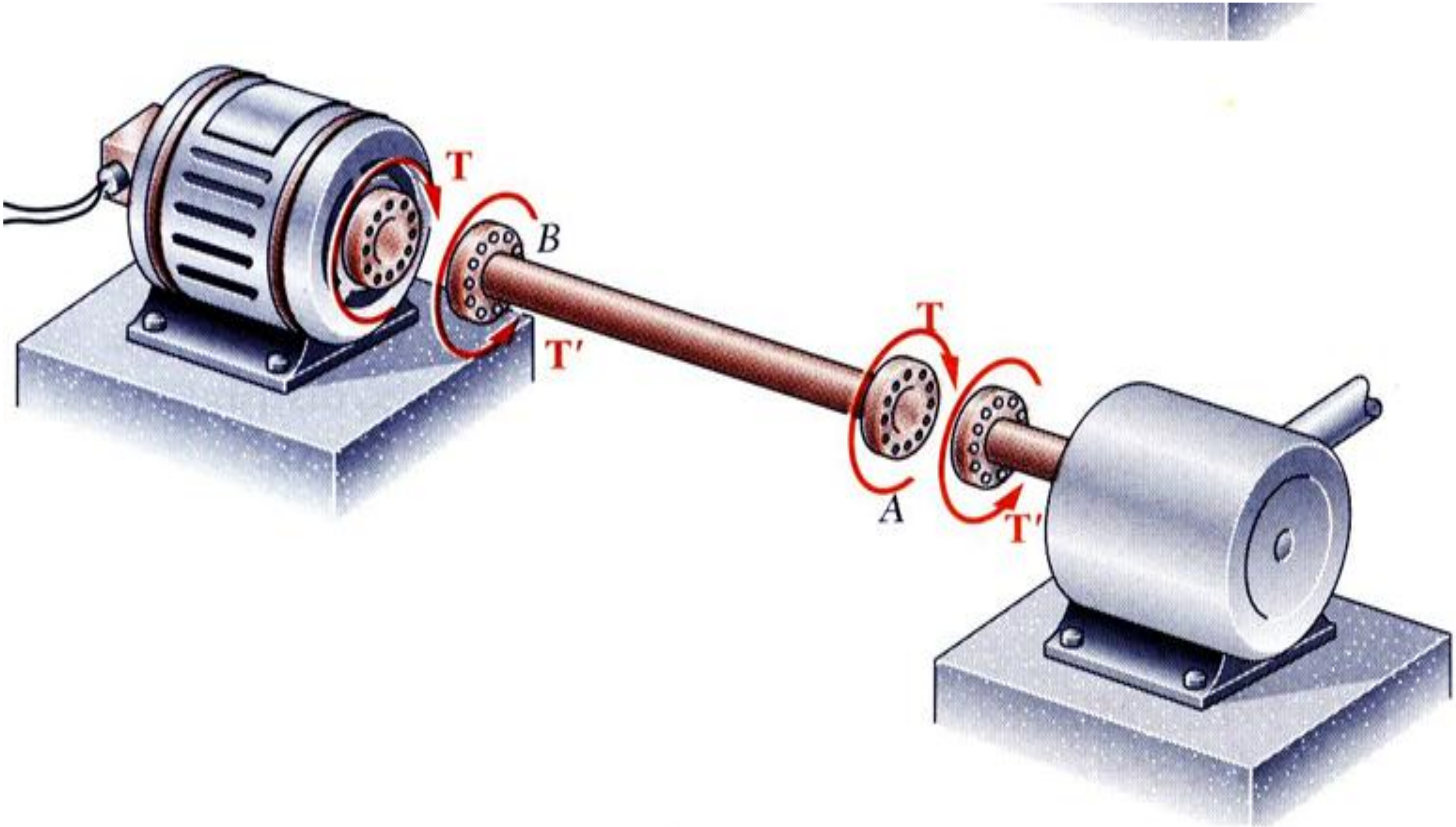


Transmission of power through a circular shaft.



Turbine exerts torque *T* on the shaft.

Shaft transmits the torque to the generator.

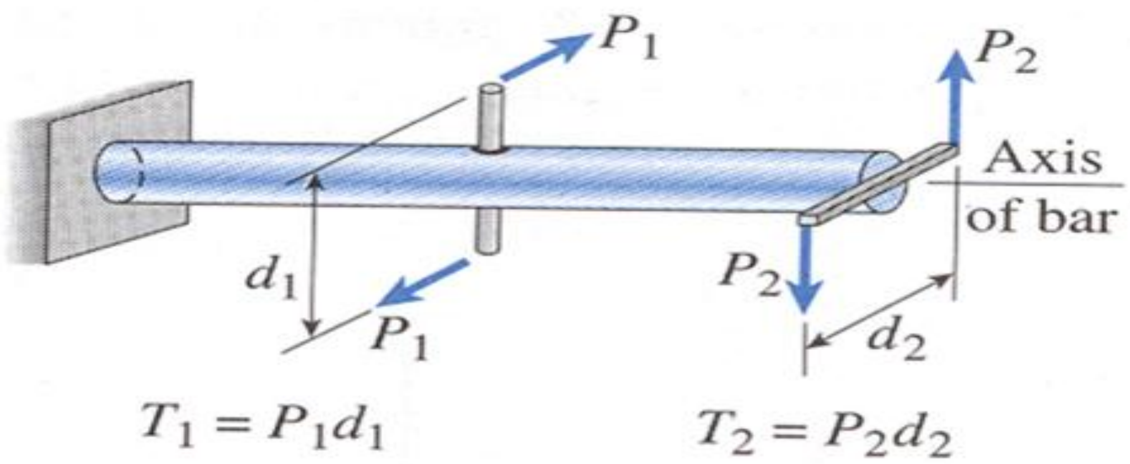


Generator creates an equal and opposite torque T.

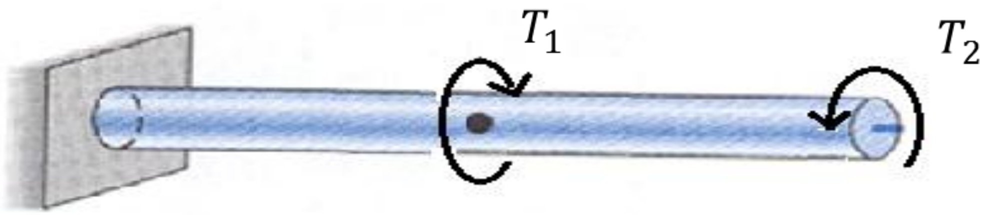
Bars subjected to Torsion

Let us now consider a straight bar supported at one end and acted upon by two pairs of equal and opposite forces as shown in Fig..

Bars subjected to Torsion



(a)



(b)

Bars subjected to Torsion

Then each pair of forces form a couple that tend to twist the bar about its longitudinal axis, thus producing surface tractions and moments.

Then we can write the moments as, $T_1 = P_1 d_1$ and $T_2 = P_2 d_2$

Definition of Torsion

When a pair of forces of equal magnitude but opposite directions acting on body, it tends to twist the body. It is known as twisting Moment or torsional moment or simply as torque.

In the case of a circular shaft if the force is applied tangentially and in the plane of transverse cross section, the torque may be calculated by multiplying the force with the radius of the shaft.

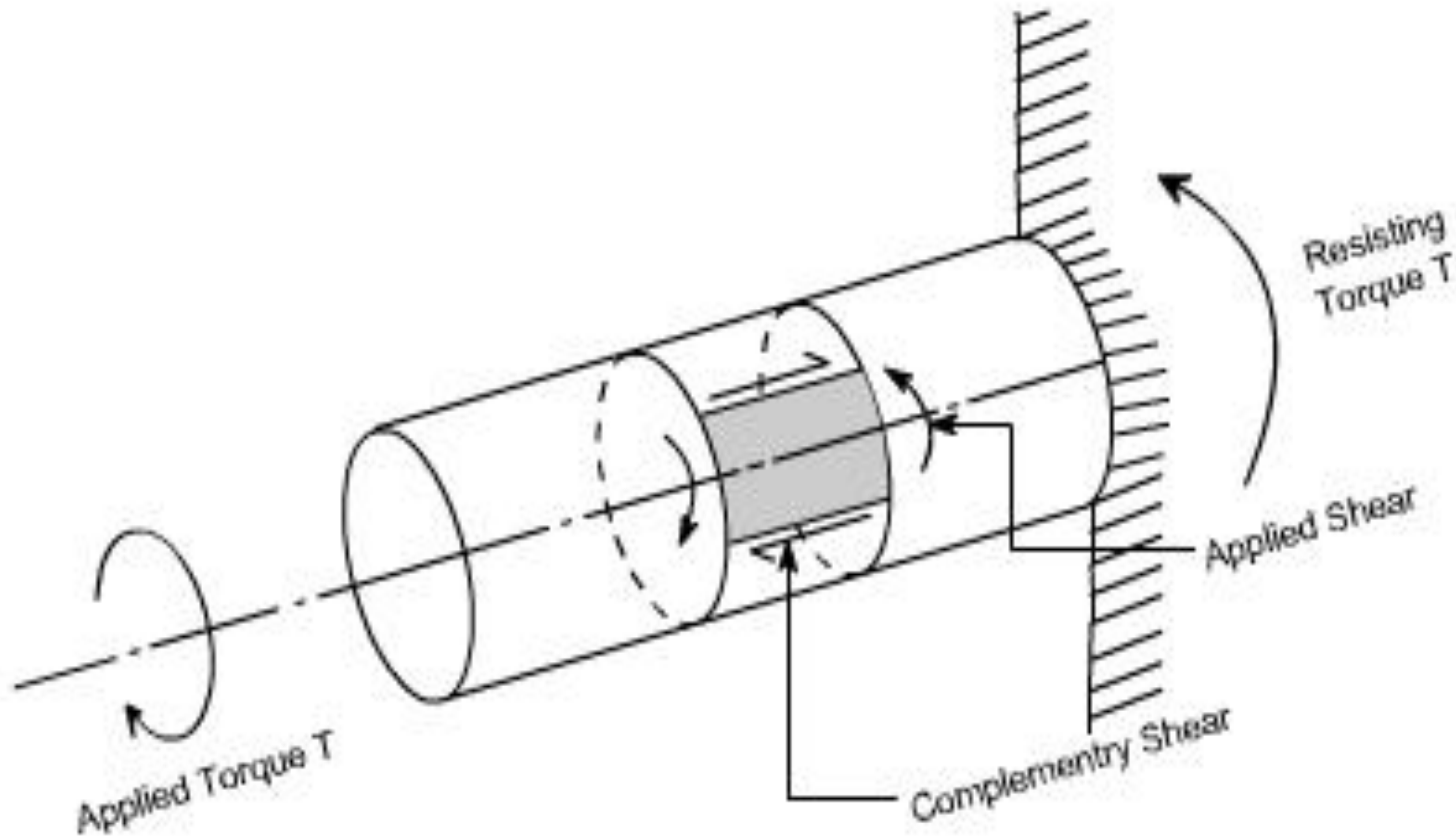
PURE TORSION

A member is said to be in pure torsion when its cross sections are subjected to only torsional moments and not accompanied by axial forces or bending moment.

SIMPLE TORSION THEORY

When a uniform circular shaft is subjected to a torque, it can be shown that every section of the shaft is subjected to a state of pure shear The moment of resistance developed by the shear stresses being everywhere equal to the magnitude, and opposite in sense, to the applied torque.

Shear System Set Up on an Element in the Surface of a Shaft Subjected to Torsion



For the purposes of deriving a simple theory to describe the behaviour of shafts subjected to torque it is necessary to make the following basic assumptions:

Assumptions Made in Theory of Torsion

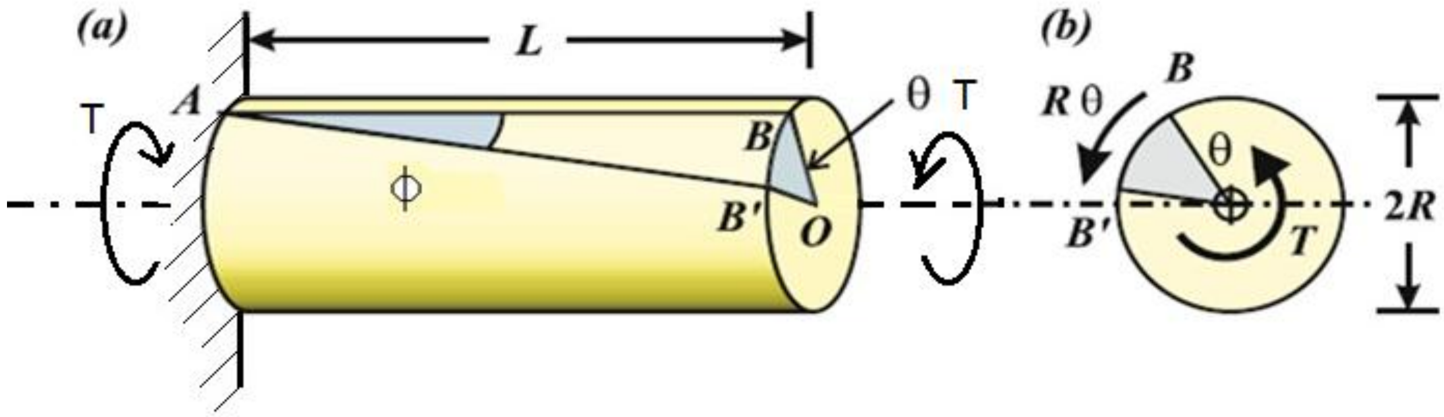
1. The material of the shaft is homogeneous, perfectly elastic and obeys Hooke's law.
2. Twist is uniform along the length of the shaft
3. The shear stress induced does not exceed the limit of proportionality.
4. Strain and deformations are small.

Assumptions Made in Theory of Torsion

5. The sections which were plane and circular before twisting remain plane and circular after twisting . (This is certainly not the case with the torsion of non-circular sections.)
6. Cross-sections rotate as it rigid, i.e. every diameter rotates through the same angle. *i.e.,* Radial lines remain radial even after applying torsional moment.

- Practical tests carried out on circular shafts have shown that the theory developed below on the basis of these assumptions shows excellent correlation with experimental results.

- **Derivation of Torsional Equations**
- Consider a shaft of length *L,* radius *R* fixed at one end and subjected to a torque *T at* the other end as shown in Fig.



Let '*O*' be the centre of circular section and *B* a point on surface.

AB be the line on the shaft parallel to the axis of shaft.

Due to torque T applied let *B* move to *B'*.

If '$\emptyset'$,shear strain $(<BAB')$ and 'θ' is the angle of twist in length *L*, then

$$R\ \theta = BB' = L\ \emptyset$$

If q_s is the shear stress at the surface of the shaft and G is modulus of rigidity then,

$$\phi = \frac{q_s}{G}$$

$$\therefore \quad R\theta = L\,\frac{q_s}{G}$$

or $$\frac{q_s}{R} = \frac{G\theta}{L}$$

Similarly if the point *B* considered is at any distance '*r*' from centre instead of on the surface, it can be shown that

$$\frac{q}{r} = \frac{G\theta}{L} \qquad \therefore \frac{q_s}{R} = \frac{q}{r} \quad \text{(or)}$$

$$q(r) = \frac{r}{R} q_s \quad (or) \quad q(r) = \frac{r}{R} q_{max}$$

Thus shear stress increases linearly from zero at axis to the maximum value q_s at surface.

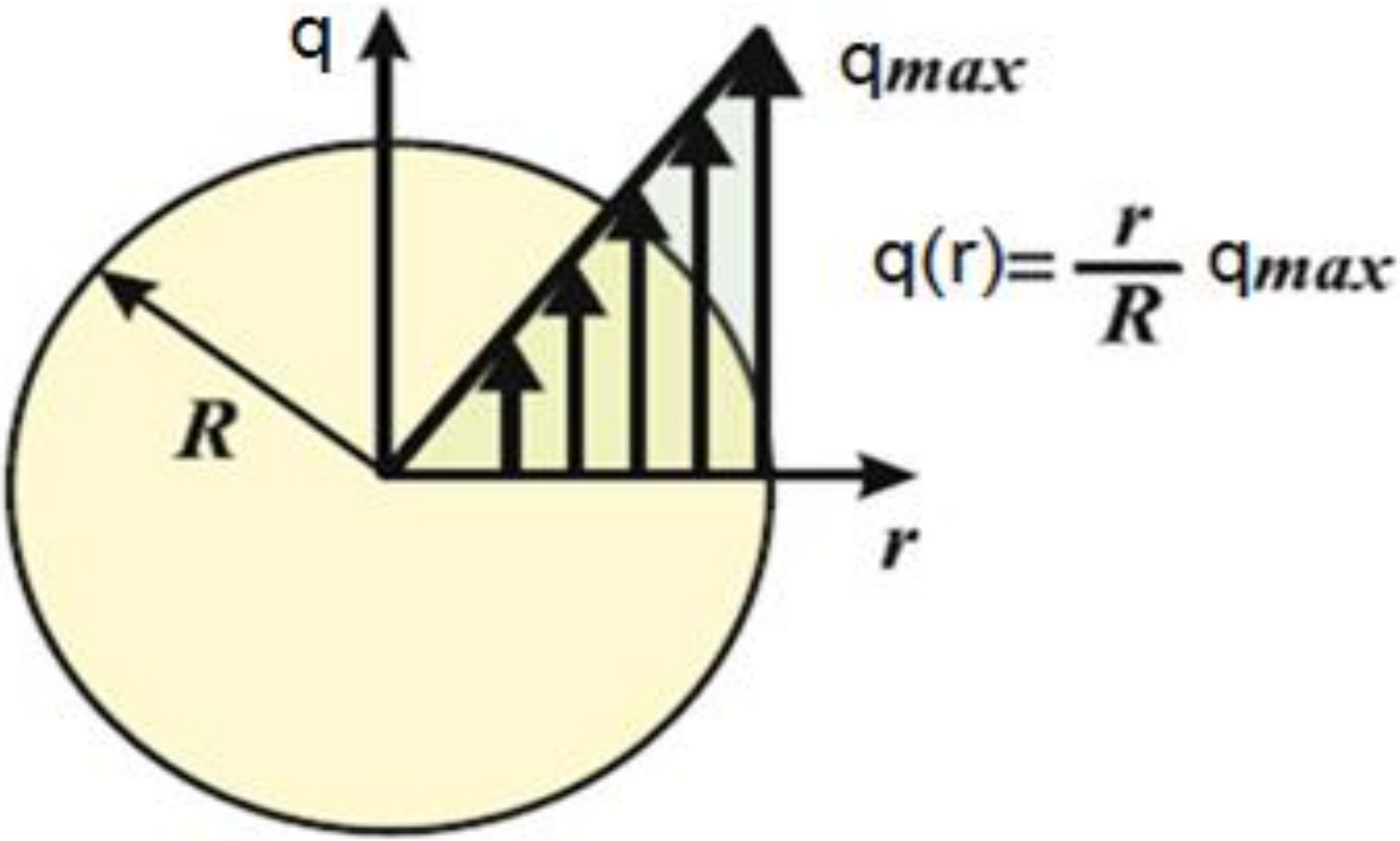
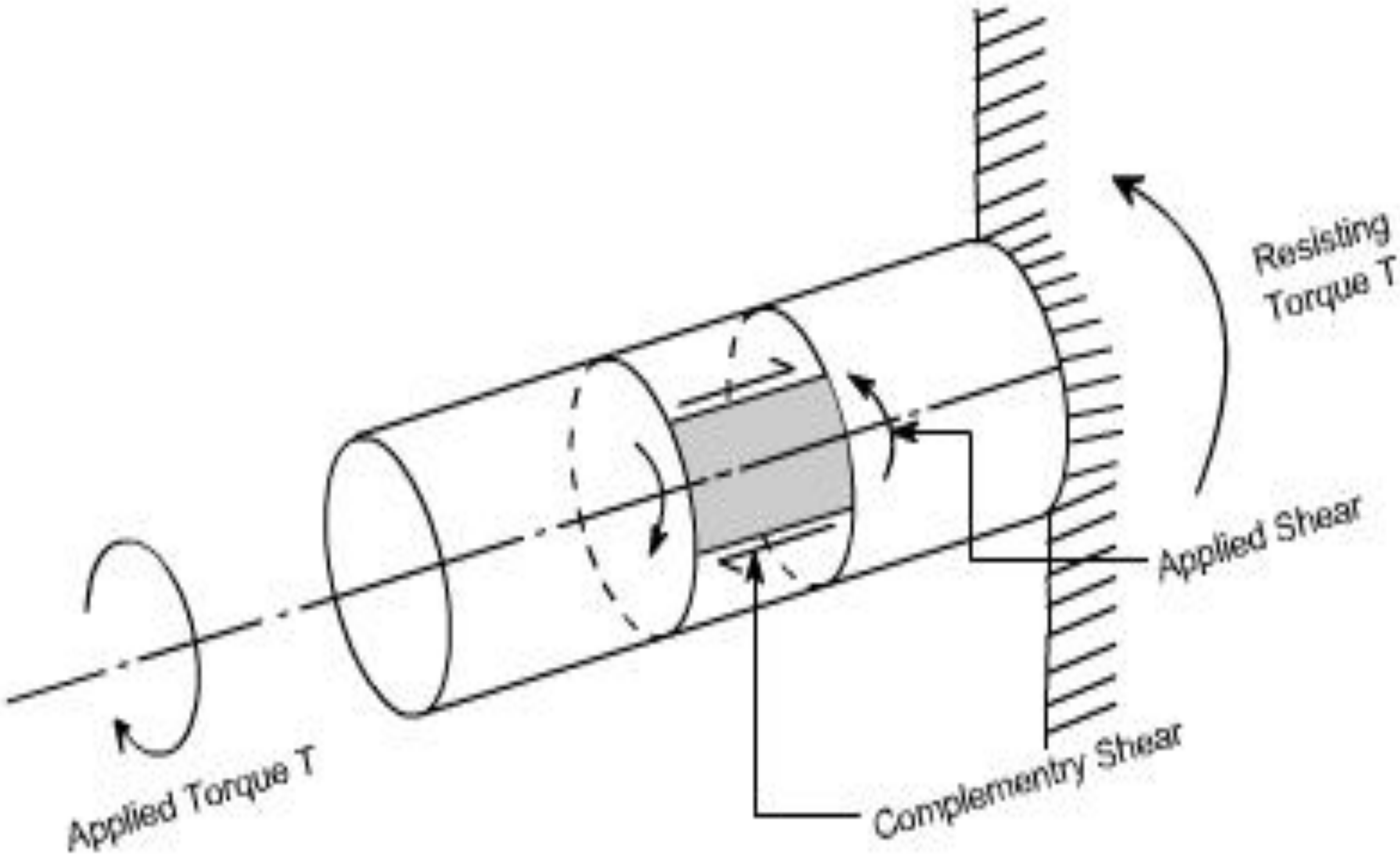
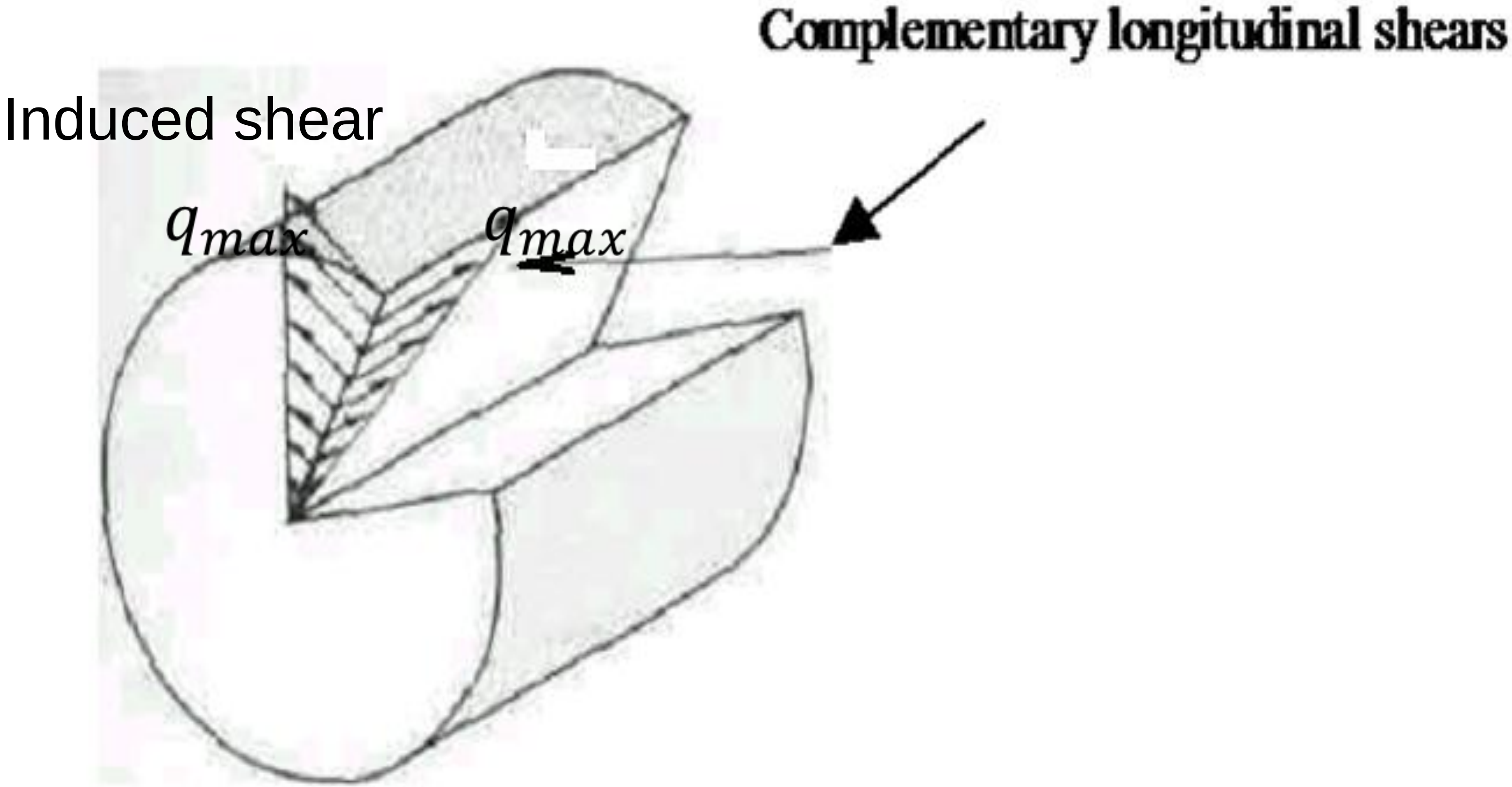


Fig.6.11 shear stress distribution in a solid shaft in torsion

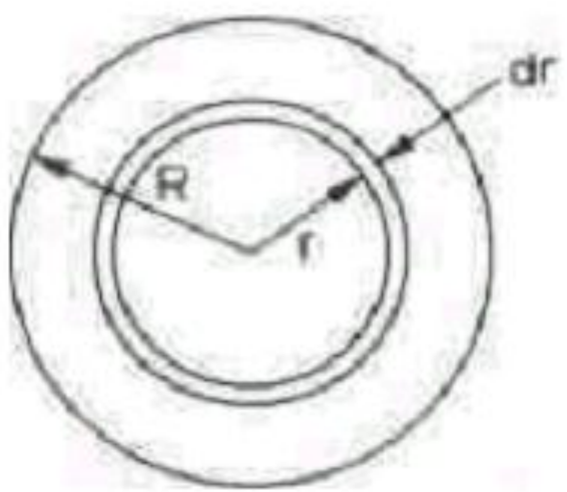

Resisting
Torque T
Applied Shear
Complementry Shear
Applied Torque T



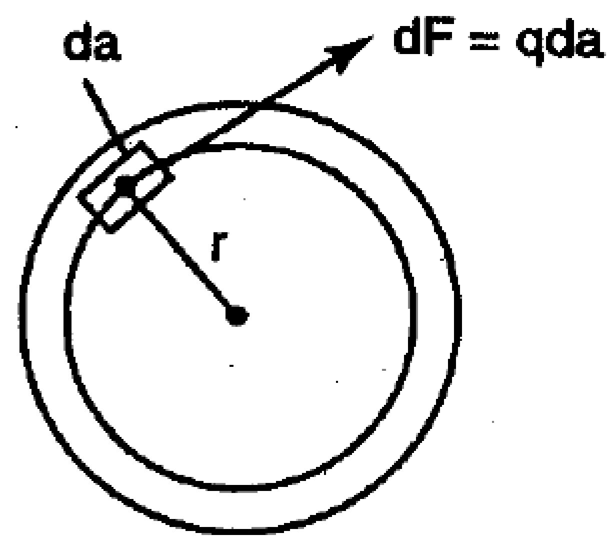
Complementary longitudinal shear stress in a shaft subjected to torsion

Strength of a solid shaft

- consider the torsional resistance developed by an elemental area *'da'* at a distance *'r'* from centre (Ref. Fig.).



(a)



(b)

If *'q'* is the shear stress developed in the element then the resisting force is $dF=qda$

$\therefore$ Resisting torsional moment due to the element,

$$dT = dF \times r$$

$$= qrda$$

We know,

$$q = q_s \frac{r}{R}$$

$$dT = q_s \frac{r^2}{R} da$$

∴ Total resisting torsional moment,

$$T = \sum q_s \frac{r^2}{R} da$$

$$= \frac{q_s}{R} \sum r^2 da$$

$$T = \frac{q_s}{R} \mathrm{J}$$

$$\frac{T}{J} = \frac{q_s}{r}$$

Where J = $\sum r^2 da$ (polar moment of inertia)

We know, $\frac{q_s}{R} = \frac{q}{r}$

$$\therefore \frac{T}{J} = \frac{q}{r} = \frac{G\theta}{L}$$ (Torsion Equation)

Where, T - torsional moment

J - polar moment of inertia

q - shear stress in the element

r- distance of element from centre of shaft

G - modulus of rigidity

θ - angle of twist

and *L* - length of shaft.

POLAR MODULUS

From torsion equation, $\frac{T}{J} = \frac{q}{r}$

But, $\frac{q}{r} = \frac{q_s}{R}$

where q_s is maximum shear stress (occurring at surface) and R is extreme fibre distance from centre

- **POLAR MODULUS**

$$\therefore \qquad \frac{T}{J} = \frac{q_s}{R}$$

$$\text{or} \qquad T = \frac{J}{R} q_s = Z_p q_s$$

where Z_p is called 'Polar Modulus of Section' .

- It may be observed that Z_p , property of the section and may be defined as the ratio of polar moment of inertia to the extreme radial distance of the fibre from the centre.

(i) For solid circular section of diameter d,

$$J = \frac{\pi}{32} d^4$$

and $$R = \frac{d}{2}$$

$$Z_p = \frac{J}{R} = \frac{\pi}{16} d^3$$

(il) For hollow circular shaft with external diameter d_1 and internal $d_{2,}$

$$J = \frac{\pi}{32}\left({d_1}^4 - {d_2}^4\right)$$

$$R = \frac{d_1}{2}$$

$$\therefore \qquad Z_p = \frac{J}{R} = \frac{\pi}{16}\frac{\left({d_1}^4 - {d_2}^4\right)}{d_1}$$

POWER TRANSMITTED

- Power is defined as the rate of doing work
- Consider a shaft subjected to a torque *T* and rotating at *N* revolutions per minute (rpm).
- Taking second as the unit of time then, angle through which torque moves is

$$= \frac{N}{60} \times 2\pi = \frac{2\pi N}{60}$$

power transmitted

Power = Work done per second $= T \times \frac{2\pi N}{60}$

$$= \frac{2\pi NT}{60}$$

(or) Power = force x distance

$$= W \times \frac{2\pi RN}{60} \qquad = WR \times \frac{2\pi N}{60}$$

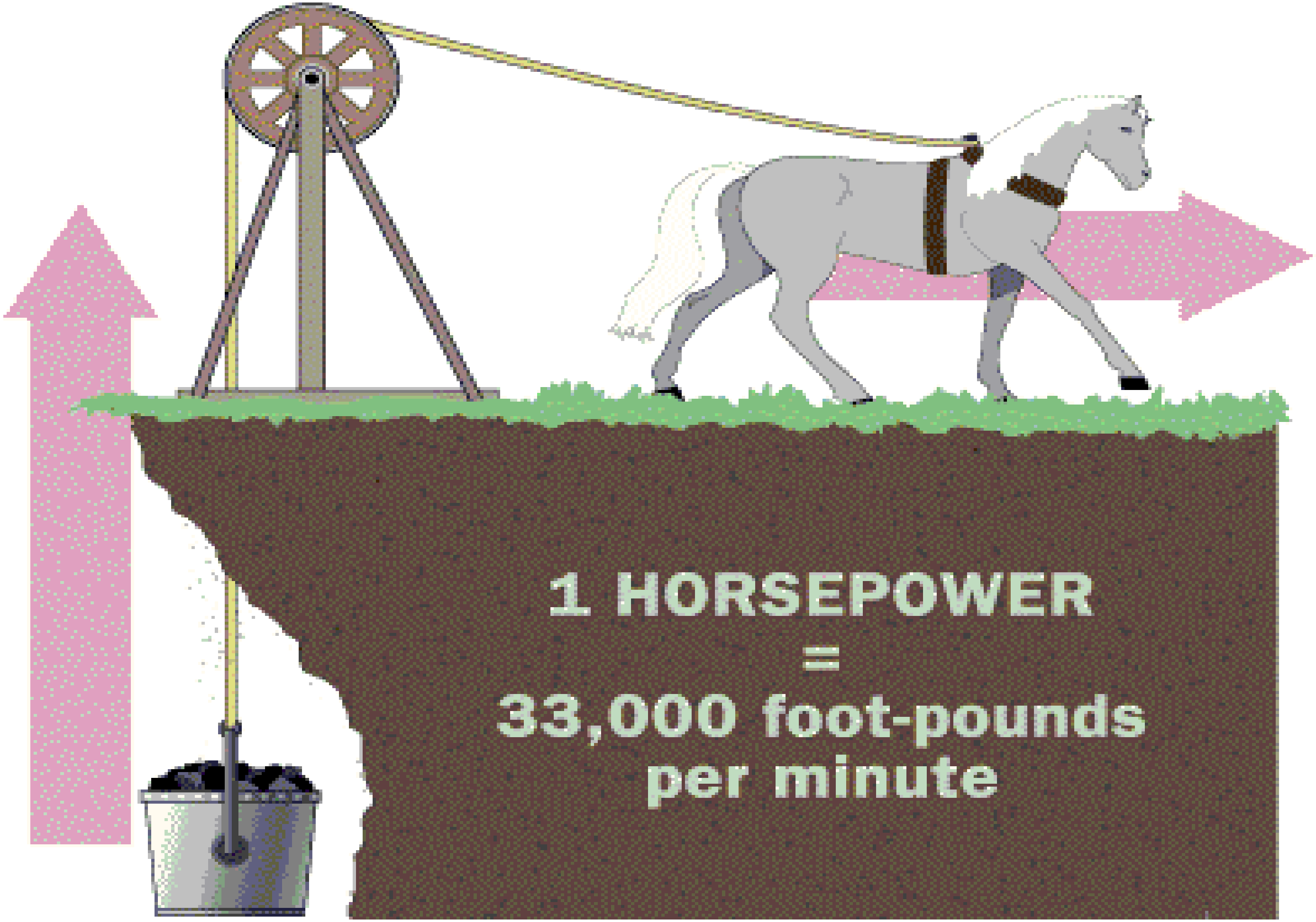
$$P = \frac{2\pi NT}{60}$$, where T = W × R

Unit for Power

- If *T* is taken in N-m, then unit of power is in N-m/sec *i.e.* Watt.
- Since Watt is a small quantity in practice, it is expressed in kilo watts (kW).
- Old practice was to express power in Horse Power unit. It may be noted that
- One Horse Power(HP) = 0.746kW

  = 746 Watts

  = 746 N m/sec.


1 HORSEPOWER
=
33,000 foot-pounds
per minute

i.e., 1 HP = 33 000 foot pounds per minute

$$= \frac{33000 \times 0.3048 \times 0.4536 \times 9.81}{60}$$

$$= 746 \text{ Nm/sec}$$

TORSIONAL RIGIDITY/STIFFNESS OF SHAFTS

From the torsion equation, $\frac{T}{J} = \frac{G\theta}{L}$

we get, $\mathrm{T} = \frac{GJ\theta}{L}$

- Hence the term *GJ* may be looked as torque required to introduce unit angle of twist in unit length, and is called torsional rigidity or stiffness of shaft.
- This term is analogous to the term flexural rigidity *(EI)* in theory of bending.

Ex. 6.1 Calculate the maximum intensity of shear stress induced and the angle of twist produced in degrees in solid shaft of 100 mm diameter, 10m long, transmitting 120 kW at 150 rpm. Take G = 82 kN/mm^2

$$P = \frac{2\pi NT}{60}$$

$$120 \times 10^6 = \frac{2\pi \times 150 \times T}{60}$$

$$T = 7.6394 \times 10^6 \text{ Nmm}$$

Using the formula, $\frac{T}{J} = \frac{q_s}{R}$

$$J = \frac{\pi}{32} d^4 = \frac{\pi}{32} \times 100^4 = 9.8175 \times 10^6 \ mm^4$$

$$\therefore \frac{7.6394 \times 10^6}{9.8175 \times 10^6} = \frac{q_s}{50}$$

$$\therefore q_s = 38.91 \ N/mm^2$$

- Angle of twist, $\theta = \frac{TL}{GJ}$

$$= \frac{7.6394 \times 10^{6} \times 10000}{82 \times 10^{3} \times 9.8175 \times 10^{6}}$$

$$= 94.895 \times 10^{-3}\ radians$$

$$= 94.895 \times 10^{-3} \times \frac{180}{\pi}\ deg$$

$$= 5.437°$$

- Ex 6.2 Determine the diameter of solid shaft which will transmit 400 kW at 280 rpm. The angle of twist must not exceed one degree per metre length and maximum torsional shear stress is to be limited to 40 N/mm^2 . Assume G =84 kN/mm^2

$$P = \frac{2\pi NT}{60}$$

$$400 \times 10^6 = \frac{2\pi \times 280 \times T}{60}$$

$$T = 13.64185 \times 10^6 \, Nmm$$

From the consideration of maximum shear stress, $\frac{T}{J} = \frac{q_s}{R}$

i.e., $$\frac{13.64185 \times 10^6}{\frac{\pi d^4}{32}} = \frac{40}{\frac{d}{2}}$$

$$\therefore d = 120.206mm.$$

From the consideration of maximum angle of twist,

$$\frac{T}{J} = \frac{G\,\theta}{l}$$

$$i.e., \frac{13.64185\times10^6}{\frac{\pi d^4}{32}} = \frac{84000 \times \frac{\pi}{180}}{1000}$$

$$\therefore d = 98.67mm.$$

∴ Minimum diameter of the shaft to be used is 120.206mm

Ex 6.3. A hollow circular shaft 20mm thick transmits 294 kW at 200r.p.m.Determine the diameters of the shaft if shear strain due to torsion is not to exceed 8.6×10^{-4}

Take modulus of rigidity as 80GN/m^2

Solution

Let, $d_1 =$ external diameter of the hollow shaft, in mm

$d_2 =$ internal diameter of the hollow shaft, in mm

$t =$ thickness of the shaft = 20 mm

Then,

$d_1 - d_2 = 2\mathrm{t} = 40\mathrm{mm}$

$[\mathrm{or}\ \ d_2 = (d_1 - 40)\mathrm{mm}]$

GIVEN:

Shear strain due to torsion,

$$e_s = 8.6 \times 10^{-4}$$

Modulus of rigidity, G $= 80 \ \times 10^3 N/mm^2$

Power transmitted,

$$P = 294kW$$

= 294 000 Nm/sec

$$= 294 \ \times \ 10^6 \ {}^{Nmm}/_{sec}$$

Speed, $N = 200\, r.p.m$

Diameters of the shaft, d_1 and d_2:

We know that, $P = \frac{2\pi NT}{60}$

$$294 \times \ 10^6 = \frac{2\pi \times 200 \times T}{60} \ \text{ or}$$

$$\mathrm{T} = \frac{294 \times \ 10^6 \times 60}{2\pi \times 200}$$

$$= 14.037 \ \times 10^6 \ Nmm$$

$$\frac{T}{J} = \frac{q_s}{R} \quad or \quad q_s = \frac{TR}{J} = \frac{14.037 \ \times 10^6 \times (\frac{d_1}{2})}{\frac{\pi}{32}({d_1}^4 - {d_2}^4)}$$

$$= \frac{71.49 \times 10^6 \, d_1}{{d_1}^4 - (\, d_1 - 40)^4} N/mm^2$$

We know, $e_s = \frac{q_s}{G}$ $\quad or \quad$ $q_s = e_s \times G$

$$= 8.6 \times 10^{-4} \times 80 \times 10^3 N/mm^2$$

$$= 68.8 N/mm^2$$

$$\therefore \frac{71.49 \times 10^6 \, d_1}{{d_1}^4 - (\, d_1 - 40)^4} = 68.8$$

or,

$$\frac{{d_1}^4 - (\, d_1 - 40)^4}{d_1} = \frac{71.49 \times 10^6}{68.8}$$

$$= 1.039 \times 10^6$$

By trail and error, $d_1 = 108mm$

And, $d_2 = 108 - 2 \times 20 = 68 \, mm$

- Ex. 6.4. During tests on a sample of steel bar 25 mm in diameter, it is found that the pull of 60 kN produces an extension of 0.1164 mm on a length of 200mm and a torque 220 N-m produces an angular twist of one degree on a length of 250mm.Find the Poisson's ratio of the steel.

Solution.

Given :

<u>Tension test data:</u>

dia., d = 25mm,

pull , P = $60 \times 10^3\ N$

extension, Δ = 0.1164mm on a length of 200mm

$$\therefore E = \frac{PL}{A\Delta} = \frac{60 \times 10^3 \times 200}{\frac{\pi \times 25^2}{4} \times 0.1164} = 210019\ N/mm^2$$

<u>Torsion test data:</u>

Torque = 220 Nm

Twist = 1° $over\ a\ length\ of\ 250mm$

From torsion formula, $\frac{T}{J} = \frac{G\theta}{l}$

$$G = \frac{Tl}{J\theta} = \frac{220\,000 \times 250}{\frac{\pi \times 25^4}{32} \times \frac{\pi}{180}} = 82172.3\ N/mm^2$$

We know,

$$E = 2G(1 + \mu)$$

$$i.e., \; 210019 = 2 \times 82172.3(1 + \mu)$$

$$\therefore \; \mu = 0.28$$

Ex. 6.5 A shaft is required to transmit 300kW power at 220 rpm. The maximum torque may be 1.5 times the mean torque. The shear stress in the shaft should not to exceed 45N/mm^2 and the twist 1°per metre length. Determine the diameter required if

a) the shaft is solid.

b) The shaft is hollow with external diameter twice the internal diameter.

Take modulus of rigidity = 80 kN/mm^2

$$P = \frac{2\pi NT}{60}$$

$$300 \times 10^6 = \frac{2\pi \times 220 \times T}{60}$$

$$T = 13.0218 \times 10^6\ Nmm$$

$$T_{max} = 1.5 \times T$$

$$= 19.53265 \times 10^6\ Nmm$$

For **solid shaft:** $J = \frac{\pi}{32} d^4$

where d is the diameter of the shaft.

From the consideration of shear stress,

$$\frac{T}{J} = \frac{q_s}{d/2}$$

$$T = \frac{\pi}{32} d^4 \times \frac{q_s}{d/2} = \frac{\pi}{16} d^3 q_s$$

Designing for maximum torque, we have

$$19.53265 \times 10^6 = \frac{\pi}{16} d^3 \times 45$$

$$d = 130.27\ mm$$

From consideration of angle of twist,

$$\frac{T_{max}}{J} = \frac{G\ \theta}{L}$$

$$T_{max} = GJ\frac{\theta}{L}$$

$$19.53265 \times 10^6 = 80 \times 10^3 \times \frac{\pi}{32}d^4 \times \frac{\pi}{180} \times \frac{1}{1000}$$

$$\therefore \quad d = 109.26\,mm$$

Use a minimum of 130.27 mm as diameter

For **Hollow Shaft:**

Let d_1 = outer diameter

d_2 = inner diameter

$= 0.5\ d_1$

$$J = \frac{\pi}{32}\left({d_1}^4 - (0.5d_1)^4\right)$$

$$= 0.092039{d_1}^4$$

From the consideration of maximum shear stress,

$$\frac{T_{max}}{J} = \frac{q_s}{d_1/2}$$

i.e., $$\frac{19.53265\times10^6}{0.092039{d_1}^4} = \frac{2\times45}{d_1}$$

$$d_1 = 133.1\,mm$$

From the consideration of angle of twist,

$$\frac{T_{max}}{J} = \frac{G\,\theta}{L}$$

$$\frac{19.53265 \times 10^6}{0.092039{d_1}^4} = 80 \times 10^3 \times \frac{\pi}{180} \times \frac{1}{1000}$$

$$d_1 = 111.03\,mm$$

Minimum dimensions to be used are

$d_1 = 133.1\,mm$ and $d_2 = 0.5d_1 = 66.55\,mm$

- Ex. 43 Compare the weight of a solid shaft with that of a hollow one having same length to transmit a given power at a given speed, if the material used for the shafts is the same. Take the inside diameter of the hollow shaft as 0.6 times the outer diameter.

- soln

Let d be the diameter of solid shaft

d_1 the outer diameter of hollow shaft

$\therefore$ Inner diameter of hollow shaft = 0.6 d_1

Let P =Power to be transmitted

N = r.p.m.

and T = the corresponding torque.

From the consideration of maximum shear stress:

Let q_s be shear at surface, $\frac{T}{J} = \frac{q_s}{R}$

or $$T = J\frac{q_s}{R}$$

Solid shaft:

$$J = \frac{\pi}{32}d^4$$

$$T = \frac{\pi}{32}d^4 \times \frac{q_s}{d/2} = \frac{\pi}{16}d^3 q_s$$

For hollow shaft, $J = \frac{\pi}{32} {d_1}^4 (1 - 0.6^4)$

$$T = \frac{\pi {d_1}^4 (1 - 0.6^4)}{32} \times \frac{q_s}{d_1/2}$$

$$= \frac{\pi}{16} \times 0.8704 \times {d_1}^3 q_s$$

$$\therefore d^3 = 0.8704\, {d_1}^3$$

$$d = 0.95479\, d_1$$

$\therefore$ Area of solid shaft $= A_s$

$$= \frac{\pi}{4} \times d^2 = \frac{\pi}{4} \times 0.9116\, {d_1}^2$$

Area of hollow shaft $= A_h = \frac{\pi}{4}\left({d_1}^2 - {d_2}^2\right)$

$$= \frac{\pi}{4}(1 - 0.6^2){d_1}^2$$

$$= \frac{\pi}{4} \times 0.64\, {d_1}^2$$

$$\therefore \qquad \frac{A_s}{A_h} = \frac{0.9116}{0.64} = 1.4244$$

From the consideration of maximum angle of twist:

$$\frac{T}{J} = \frac{G\,\theta}{L} \quad \text{or} \quad \mathrm{T} = GJ\,\frac{\theta}{L}$$

$\therefore$ For solid shaft, $T = \dfrac{\pi}{32} d^4 \dfrac{G\,\theta}{L}$

For hollow shaft, $T = \dfrac{\pi}{32}\left({d_1}^4 - (0.6d_1)^4\right)\dfrac{G\,\theta}{L}$

$$d^4 = {d_1}^4(1 - 0.6^4) = 0.8704{d_1}^4$$

$$d^2 = 0.93295\ {d_1}^2$$

Now, Area of solid shaft $= \frac{\pi}{4} \times d^2$

$$= \frac{\pi}{4} \times 0.93295 {d_1}^2$$

and Area of hollow shaft $= \frac{\pi}{4} \times \left({d_1}^2 - {d_2}^2\right)$

$$= \frac{\pi}{4} \times (1 - 0.6^2) {d_1}^2$$

$$= \frac{\pi}{4} \times 0.64\, {d_1}^2$$

Since weight is proportional to cross-sectional area when lengths are same,

$$\frac{Weight\ of\ solid\ shaft}{Weight\ of\ hollow\ shaft} = \frac{A_s}{A_h}$$

$$= \frac{0.93295}{0.64}$$

$$= 1.4577$$

Thus the weight of solid shaft is 42.44% more than that of hollow shaft if shear stress governs the design and it is 45.77% more if angle of twist governs the design.

- Ex. 44. Prove that a hollow shaft is stronger and stiffer than the solid shaft of the same material, length and weight.
- *Solution*
- Let d be the diameter of solid shaft
- d_1 be the outer diameter of hollow shaft and d_2 inner diameter of hollow shaft.
- The two shafts have equal weight and length and are of same material.
- Hence, equating the weight of solid shaft to that of hollow shaft, we get,

$$\frac{\pi}{4} \times d^2 \rho L = \frac{\pi}{4} \times \left({d_1}^2 - {d_2}^2\right) \rho L$$

Where ρ – unit weight and L – length

i.e., $$d^2 = \left({d_1}^2 - {d_2}^2\right)$$

To prove hollow shaft is sronger:

Let T_s be the torque resisting capacity of solid shaft and T_h that of hollow shaft.

Then from torsion equation,

$$T_s = J_s \times \frac{q_s}{d/_2} = \frac{\pi d^4}{32} \times \frac{q_s}{d/_2} = \frac{\pi d^3}{16} q_s$$

and $$T_h = J_h \frac{q_s}{d_1/_2} = \frac{\pi}{32} \times \left({d_1}^4 - {d_2}^4\right) q_s \frac{2}{d_1}$$

$$= \frac{\pi}{16} \frac{\left({d_1}^4 - {d_2}^4\right)}{d_1} q_s$$

$$\therefore \qquad \frac{T_h}{T_s} = \frac{\left({d_1}^4 - {d_2}^4\right)}{d^3 d_1}$$

$$\frac{T_h}{T_s} = \frac{\left({d_1}^2 - {d_2}^2\right)\left({d_1}^2 + {d_2}^2\right)}{d_1\left({d_1}^2 - {d_2}^2\right)^{3/2}}$$

$$= \frac{\left({d_1}^2 + {d_2}^2\right)}{d_1\sqrt{\left({d_1}^2 - {d_2}^2\right)}}$$

$$= \frac{\left({d_1}^2 + {d_2}^2\right)}{{d_1}^2\sqrt{\left(1 - [\frac{d_2}{d_1}]^2\ \right)}}$$

$$\frac{T_h}{T_s} = \frac{1 + (\frac{d_2}{d_1})^2}{\sqrt{1 - (\frac{d_2}{d_1})^2}}$$

Thus numerator > 1 and denominator < 1

$$T_h > T_s$$

i.e., hollow shaft is stronger than solid shaft when their weights are same

To prove hollow shaft is stiffer:

- Stiffness of shaft may be defined as torque required to produce unit rotation in unit length.
- Let this be denoted by '*k*'. Then from torsion formula

$$\frac{k}{J} = \frac{G \times 1}{1} \qquad (or) \qquad k = GJ$$

Let k_s be the stiffness of solid shaft and

k_h be the stiffness of hollow shaft

$$k_s = G \times \frac{\pi d^4}{32}$$

$$k_h = G \times \frac{\pi}{32}\left({d_1}^4 - {d_2}^4\right)$$

$$\frac{k_h}{k_s} = \frac{\left({d_1}^4 - {d_2}^4\right)}{d^4}$$

$$\frac{k_h}{k_s} = \frac{\left({d_1}^2 - {d_2}^2\right)\left({d_1}^2 + {d_2}^2\right)}{\left({d_1}^2 - {d_2}^2\right)^2}$$

$$= \frac{\left({d_1}^2 + {d_2}^2\right)}{\left({d_1}^2 - \ {d_2}^2\right)} > 1$$

$$\therefore \quad k_h > k_s$$

i.e. Hollow shaft is stiffer than solid shaft when their weights are same.

- Ex: 45. A solid shaft transmits 350 kW at 120 rpm. If the shear stress is not to exceed *70N/mm^2, what should be the diamete*r of the shaft? If this shaft is to be replaced by a hollow one whose internal diameter = 0.6 times outer diameter, determine the size and the percentage saving in weight, the maximum shearing stress being the same.

From the relation, $P = \frac{2\pi NT}{60}$

We get, $350 \times 10^6 = \frac{2\pi \times 120 \times T}{60}$

$\therefore \quad T = 27.8521 \times 10^6 \ Nmm$

<u>For solid shaft</u>

$$T = J\frac{q_s}{R} = \frac{\pi d^4}{32}\frac{q_s}{d/_2} = \frac{\pi}{16}d^3 q_s$$

$$\therefore 27.8521 \times 10^6 = \frac{\pi}{16} d^3 \times 70$$

$$or \qquad d = 126.545\ mm$$

For hollow shaft

Let d_1 be the outer diameter and d_2 be the inner diameter of hollow shaft

$$i.e., \qquad d_2 = 0.6 d_1$$

$$\therefore \qquad T = \frac{\pi}{32}\left({d_1}^4 - (0.6d_1)^4\right)\frac{q_s}{{d_1}/_2}$$

$$27.8521 \times 10^6 = \frac{\pi}{16}(1 - 0.6^4){d_1}^3 \times 75$$

$$d_1 = 129.523\ mm$$

$$\therefore \qquad d_2 = 77.71\ mm$$

∴ Cross-sectional area of hollow shaft

$$= \frac{\pi}{4} \times \left({d_1}^2 - {d_2}^2\right) = 8433.106 mm^2$$

and cross-sectional area of solid shaft

$$= \frac{\pi}{4} \times d^2 = 12577.08\, mm^2$$

% saving in weight

$$= \frac{Weight\ of\ solid\ shaft - weight\ of\ hollow\ shaft}{weight\ of\ solid\ shaft} \times 100$$

$$= \frac{(12577.08 - 8433.106)\rho L}{12577.08 \times \rho L} \times 100$$

$$= 32.95$$

Ex. 46. What percentage of strength of a solid circular steel shaft 100 mm diameter is lost by boring 50 mm axial hole in it? Compare the strength and weight ratio of the two cases.

- ***Solution***

<u>Diameter of solid shaft</u>

d= 100mm

<u>Diameter of hollow shaft:</u>

outer, d_1 = 100mm

inner, d_2 = 50 mm

$$T_s = J\frac{q_s}{R} = \frac{\pi d^4}{32}\frac{q_s}{d/2} = \frac{\pi}{16}100^3 q_s$$

$$T_h = J\frac{q_s}{R} = \frac{\pi}{32}\left({d_1}^4 - {d_2}^4\right)\frac{q_s}{d_1/2}$$

$$= \frac{\pi}{32}\left((100)^4 - (50)^4\right)\frac{q_s}{50}$$

$$= \frac{\pi}{32}\frac{(100)^4}{50}\left(1 - (0.5)^4\right)q_s$$

$$\therefore \quad \frac{T_h}{T_s} = 1 - (0.5)^4 = 0.9375$$

$$\therefore \text{Loss in strength} = \left(\frac{T_s - T_h}{T_s}\right) \times 100$$

$$= \left(1 - \frac{T_h}{T_s}\right) \times 100$$

$$= (1 - 0.9375) \times 100$$

$$= 6.25$$

$$\text{Weight ratio} = \frac{W_h}{W_s} = \rho L \frac{A_h}{\rho L A_s}$$

$$= \frac{A_h}{A_s}$$

$$= \frac{\frac{\pi}{4}\left({d_1}^2 - {d_2}^2\right)}{\frac{\pi}{4} d^2}$$

$$= \frac{100^2 - 50^2}{100^2}$$

$$= 0.75$$

Ex:47. A solid shaft of mild steel 200mm in diameter is to be replaced by hollow shaft of alloy steel for which the allowable shear stress is 22percent greater. If the power to be transmitted is to be increased by 20 percent and the speed of rotation increased by 6 percent. Determine the maximum internal diameter of the hollow shaft. The external diameter of the hollow shaft is to be 200mm.

solution

For solid shaft :

Diameter, $D_S = 200mm\ (given)$

$$q_S = maximum\ shear\ stress\ on\ the\ surface\ of\ solid\ shaft$$

$$P_S = power\ transmitted$$

$$N_S = rate\ of\ rotation$$

$$(J)_S = polar\ moment\ of\ inertia, and$$

$$T_S = torque\ transmitted$$

For hollow shaft :

Let $q_H, P_H, N_H, (J)\ _H$ and T_H be the corresponding notations respectively for the hollow shaft

GIVEN: $q_H = 1.22 q_S$

$$P_H = 1.2 P_S$$

$$N_H = 1.06 N_S$$

Outer diameter of hollow shaft,

$$d_1 = 220\ mm$$

Maximum internal diameter of hollow shaft, d_2:

Now, $$P = \frac{2\pi NT}{60}$$

$$\therefore \; P_S = \frac{2\pi N_S T_S}{60} \text{ and } P_H = \frac{2\pi N_H T_H}{60}$$

but, $$P_H = 1.2\, P_S$$

$$\frac{2\pi N_H T_H}{60} = 1.2 \times \frac{2\pi N_s T_s}{60}$$

Or, $$(1.06N_S)T_H = 1.2(N_S)T_S$$

$$T_H = 1.132T_S$$

From the relation

$\frac{T}{J} = \frac{q}{R}$, we have

$$\frac{T_S}{(J)_S} = \frac{(q)_S}{(\frac{200}{2})} \text{ ----- } (i)$$

And $$\frac{T_H}{(J)_H} = \frac{(q)_H}{(\frac{200}{2})} \text{ ----- } (ii)$$

Dividing (i) and (ii), we get

$$\frac{T_S \times (J)_H}{T_H \times (J)_S} = \frac{(q)_s}{(q)_H}$$

Or, $$\frac{T_S \times \frac{\pi}{32}({d_1}^4 - {d_2}^4)}{1.132 T_S \times \frac{\pi}{32} {D_S}^4} = \frac{(q)_s}{1.22(q)_S}$$

$$= 0.8916$$

i.e., $$\frac{((200)^4 - {d_2}^4)}{1.132 \times (200)^4} = 0.8196$$

$$\begin{aligned} {d_2}^4 &= (200)^4 - 0.8196 \times 1.132 \times (200)^4 \\ &= 1.1554 \times 10^8 \end{aligned}$$

$$d_2 = 103.7\ mm$$

(internal diameter of the hollow shaft)

COMPOUND SHAFTS

- Compound shafts are shafts made up of a number of small shafts of different cross-sections or of different materials connected to form a composite shaft to transmit or resist the torque applied.
- **Types**

  Shafts in series

  Shafts in parallel

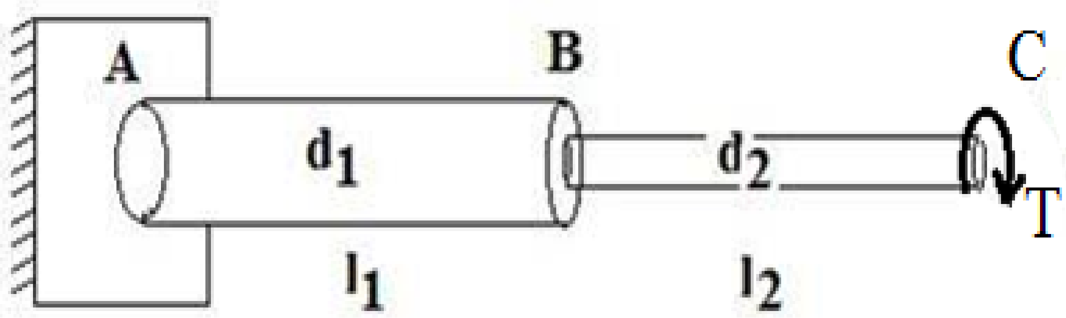
Shafts in series:

In order to form a composite shaft sometimes number of small shafts of different cross-sections or of different materials shafts are connected in series.

To analyze these shafts, first torque resisted by each portion is found and then individual effects are clubbed.

Shafts in series:

- At fixed end torque of required magnitude develops to keep the shaft in equilibrium.
- The torques developed at the ends of any portion are equal and opposite.
- The angle of twist is the sum of the angle of twist of the shafts connected in series.



$$T_1 = T_2$$

$$Total\ angle\ of\ twist, \theta = \theta_1 + \theta_2$$

$$\therefore \theta = \frac{Tl_1}{G_1J_1} + \frac{Tl_2}{G_2J_2} = T(\frac{l_1}{G_1J_1} + \frac{l_2}{G_2J_2})$$

Shafts in Parallel:

The shafts are said to be in parallel when the driving torque is applied at the junction of the shafts and the resisting torque is at the other ends of the shafts .

Here, the angle of twist is same for each shaft, but the applied torque is shared between the two shafts.

Shafts in Parallel:

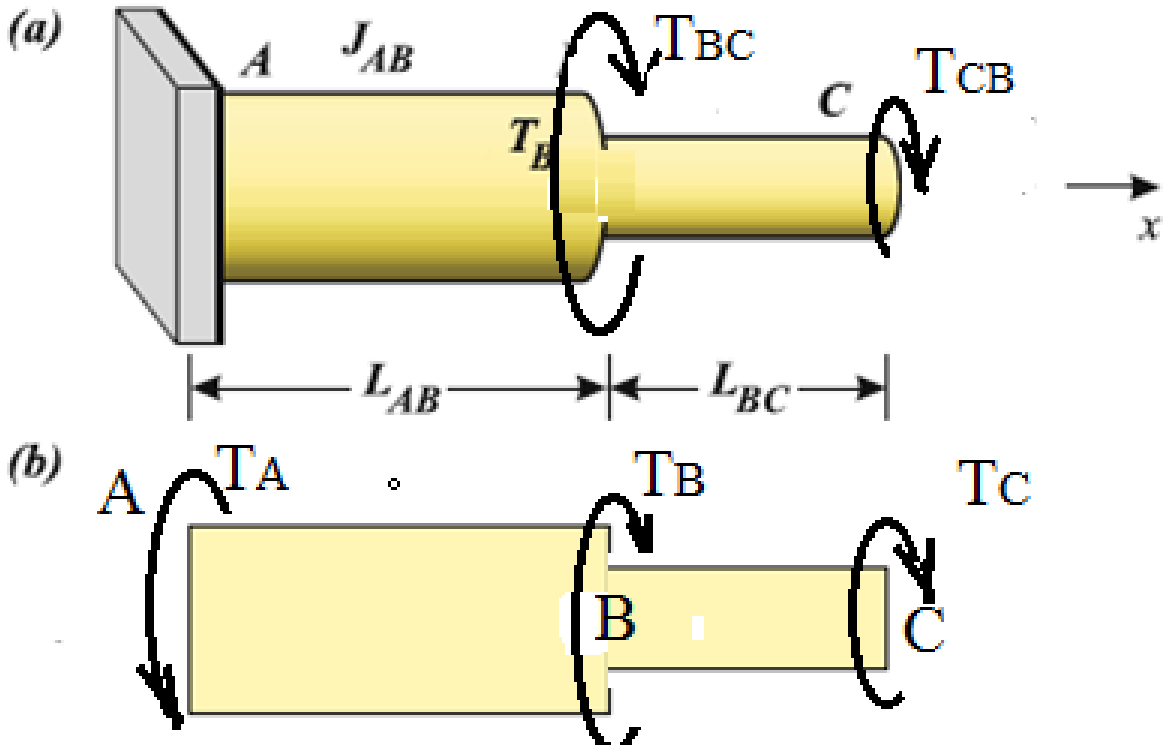
$$\text{i.e.,}\,\theta_1 = \theta_2$$

$$\frac{T_1 l_1}{G_1 J_1} = \frac{T_2 l_2}{G_2 J_2}$$

and

$$T = T_1 + T_2$$

- ***Ex :48.*** A solid stepped shaft, *ABC*, is fixed at *A*, and is subjected to torques T_B = 880 N·m and T_C = 275 N·m as shown in Fig.. Segment *AB* has a length *of* 1.5 m and diameter 50 mm. Segment *BC* has a length 1.0 m and diameter 30 mm. The material is steel with a shear modulus of *G* = 77 GPa. Determine (a) the maximum shear stress in *AB*, (b) the maximum shear stress in *BC*, and (c) the total angle of twist of shaft *ABC*, θ_{AC} .



- **Fig. (a)** A stepped-shaft subjected to torques T_B and T_C. **(b)** FBD of entire system.

- ***Step 1*. The polar moment of inertia of each cross-section**

$$J_{AB} = \frac{\pi(D_{AB})^4}{32} = \frac{\pi(50)^4}{32} = 613.6 \times 10^3 mm^4$$

$$J_{BC} = \frac{\pi(D_{\boldsymbol{BC}})^4}{32} = \frac{\pi(30)^4}{32} = 79.52 \times 10^3 mm^4$$

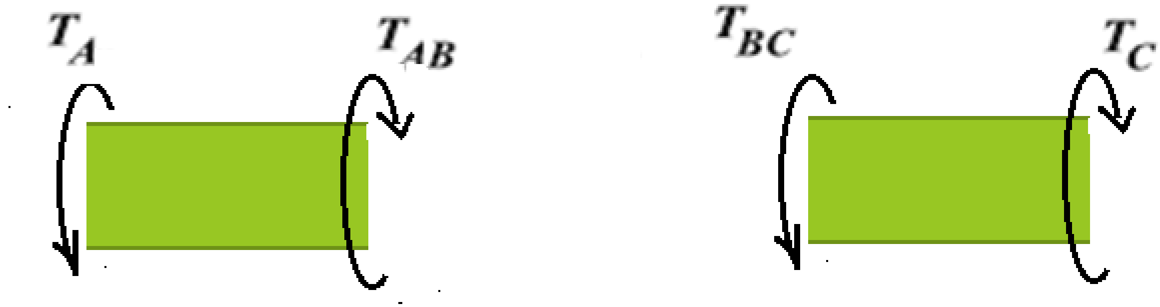


Fig.7.3 FBDs to find internal torques T_{AB} and T_{BC} .

Step 2: From the FBD of the shaft (figure7.2(b) the reaction at the wall is:

$$T_A = T_B + T_C = 880 + 275 = 1155 \text{ Nm}$$

The torques carried in AB and BC are:

$$T_{AB} = T_A = 1155Nm$$
$$T_{BC} = T_C = 275Nm$$

Step 3: The maximum shear stress and angle of twist of segment AB are:

maximum shear stress, $q_{AB,max} = \frac{T_{AB} R_{AB}}{J_{AB}}$

$$= \frac{(1155000 \times 25)}{613.6 \times 10^3}$$

$$= 47.06\ N/mm^2$$

Angle of twist of segment AB

$$\theta_{AB} = \frac{T_{AB} L_{AB}}{J_{AB} G}$$

$$= \frac{(1155\,0000 \times 1500)}{(613.6 \times 10^3)(75 \times 10^3)}$$

$$= 0.0376\, rad = 2.16^o$$

Step 4: The maximum shear stress and angle of twist of segment BC are

$$q_{BC,max} = \frac{T_{BC}R_{BC}}{J_{BC}} = \frac{(275000 \times 15)}{79.52 \times 10^3}$$

$$q_{BC,max} = 51.87\, N/mm^2$$

$$\theta_{BC} = \frac{T_{BC}L_{BC}}{J_{BC}G} = \frac{(275000 \times 1000)}{(79.52 \times 10^3)(75 \times 10^3)}$$

$$= 0.0461\, rad = 2.64\, ^o$$

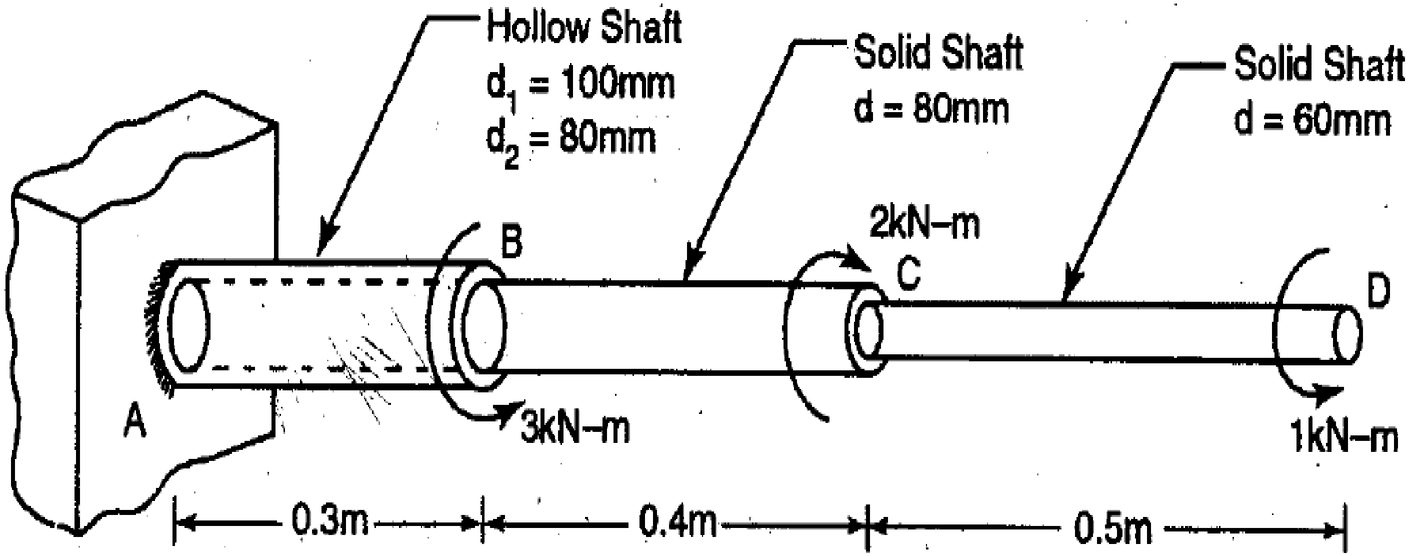
Step 5: Total angle of twist

since the internal torques twist both segments in the same direction, the total angle of twist is

$$\theta_{AC} = \theta_{AB} + \theta_{BC} = 2.16^o + 2.64^o$$

$$Answer: \quad \theta_{AC} = 4.8^o$$

Ex:49. A stepped shaft is subjected to torque as shown in Fig. Determine the angle of twist at the free end. Take G = 80 kN/mm^2 . Also find the maximum shear stress in any Step.



(a)

Fig. (b) shows the free body diagram of the portions of shaft

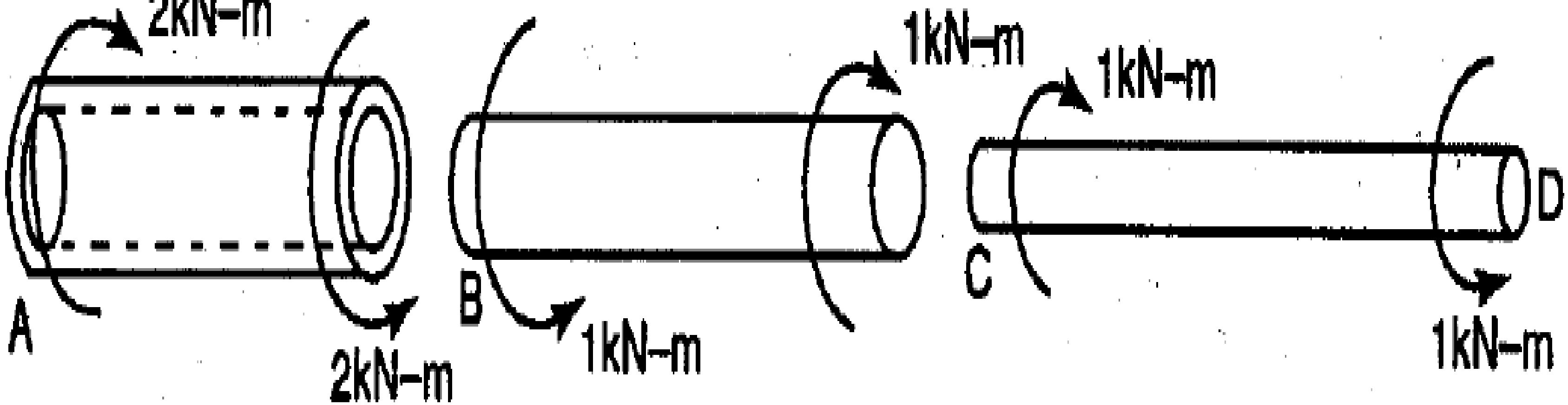


Fig (b)

Now polar modulus of section of

AB, $J_1 = \frac{\pi}{32}(100^4 - 80^4)$

$$= 5.796238 \times 10^6\, mm^4$$

BC, $J_2 = \frac{\pi}{32}(80^4) = 4.021238 \times 10^6\, mm^4$

CD, $J_3 = \frac{\pi}{32}(60^4) = 1.272345 \times 10^6\, mm^4$

From torsion formula, $\theta = \frac{TL}{GJ}$

$\therefore$ Angle of twist at free end

= Angle of twist of AB – Angle of twist of BC + Angle of twist of CD

$$= \theta_1 - \theta_2 + \theta_3$$

$\therefore$ Angle of twist at free end

$$= \frac{2\times10^6\times300}{80\times10^3\times5.796238\times10^6}$$

$$- \frac{1\times10^6\times400}{80\times10^3\times4.021238\times10^6}$$

$$+ \frac{1\times10^6\times500}{80\times10^3\times1.272345\times10^6}$$

$$= 4.96273\times10^{-3}\ \ rad.$$

From torsion equation $$\frac{T}{J} = \frac{q_s}{R}$$

Shear stress in *AB is* $$q_{s1} = \frac{2\times10^6}{5.796238\times10^6} \times 50$$

$$= 17.25\ N/mm^2$$

Shear stress in BC; $q_{s2} = \frac{1 \times 10^6}{4.021238 \times 10^6} \times 40$

$= 9.95\ N/mm^2$

Shear stress in CD; $q_{s3} = \frac{1 \times 10^6}{1.272345 \times 10^6} \times 30$

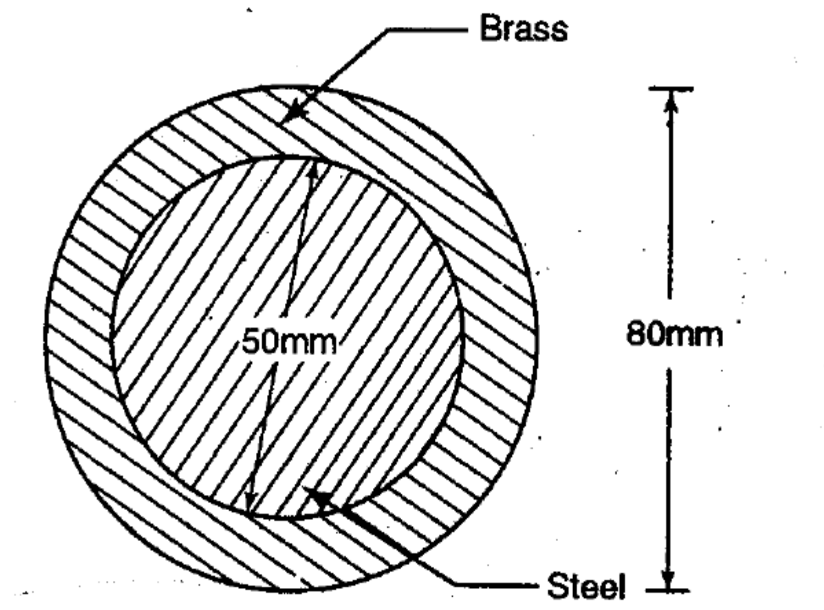
$= 23.579\ N/mm^2$

:. Maximum shear stress occurs in portion *CD* and is equal to 23.579 N/mm^2.

- Ex:50. A brass tube of external diameter 80 mm and internal diameter 50 mm is closely fitted to a steel rod of 50 mm diameter to form a composite shaft. If a torque of 10 kNm is to be resisted by this shaft, find the maximum stresses developed in each material and the angle of twist in 2 m length.

Take $G_b = 40 \times 10^3$ N/mm^2

$G_s = 80 \times 10^3 N/mm^2$



$$J_s = \frac{\pi}{32} \times 50^4 = 613592.32\ mm^4$$

$$J_b = \frac{\pi}{32} \times (80^4 - 50^4) = 3407646.3\ mm^4$$

- Let T_s, be the torque resisted by steel and

  T_b the torque resisted by brass tube.

- Then, $T_s + T_b = T$ = 10 kNm

  = 10×10^6 Nmm ----(1)

Since the angle of twist will be same in the two materials.

$$\theta_s = \theta_b$$

$$\therefore \qquad \frac{T_s L_s}{G_s J_s} = \frac{T_b L_b}{G_b J_b}$$

$$or \qquad T_s = \frac{G_s}{G_b} \times \frac{J_s}{J_b} T_b \qquad Since\ L_s = L_b$$

$$= \frac{80 \times 10^3}{40 \times 10^3} \times \frac{613592.32}{3407646.3} T_b$$

$$= 0.360\, T_b \qquad \text{-----} \qquad (2)$$

From (1) and (2) we get,

$$0.360\,T_b + T_b \;\; = 10 \times 10^6\; N - mm$$

$i.e.,$

$$T_b = 7.353 \times 10^6\; N - mm$$

$$\therefore \qquad T_s = 2.647 \times 10^6\; N - mm$$

Maximum stress in steel q_{smax} is given by

$$\frac{T_S}{J_S} = \frac{q_{smax}}{R_S}$$

$$q_{smax} = \frac{2.647 \times 10^6 \times 25}{613592.32}$$

$$= 107.85\ N/mm^2$$

Maximum stress in brass tube q_{bmax} is given by

$$\frac{T_b}{J_b} = \frac{q_{bmax}}{R_b}$$

$$q_{bmax} = \frac{7.353 \times 10^6 \times 40}{3407646.3}$$

$$= 86.312\ N/mm^2$$

Angle of twist in brass = angle of twist in steel

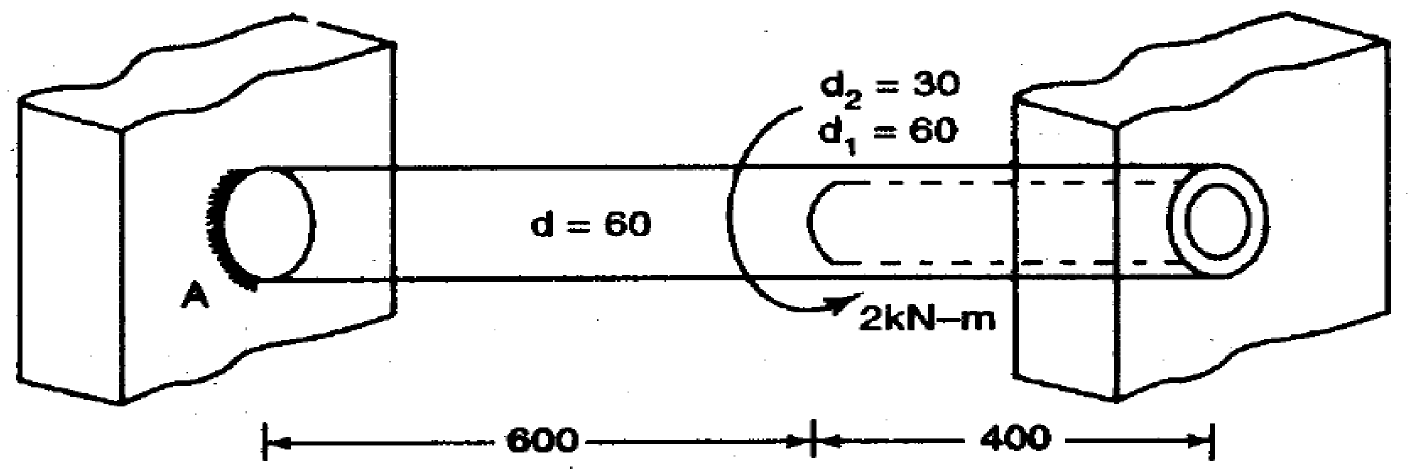
$$= \frac{T_S L_S}{G_S J_S}$$

$$= \frac{2.647 \times 10^6 \times 2000}{80 \times 10^3 \times 613592.32}$$

$$= 0.10785\ radians$$

= 6.18°

- Ex:51. A bar of length 1000 mm and diameter 60 mm is centrally bored for 400 mm, the bore diameter being 30 mm as shown in Fig. (a). If the two ends are fixed and is subjected to a torque of 2 kNm as shown in figure, find the maximum stresses developed in the two portions.



(a)

Fig.7.6(b) shows the free body diagrams.

Let T_1 be the torque resisted by portion AB and *T2 that* resisted by *BC.*

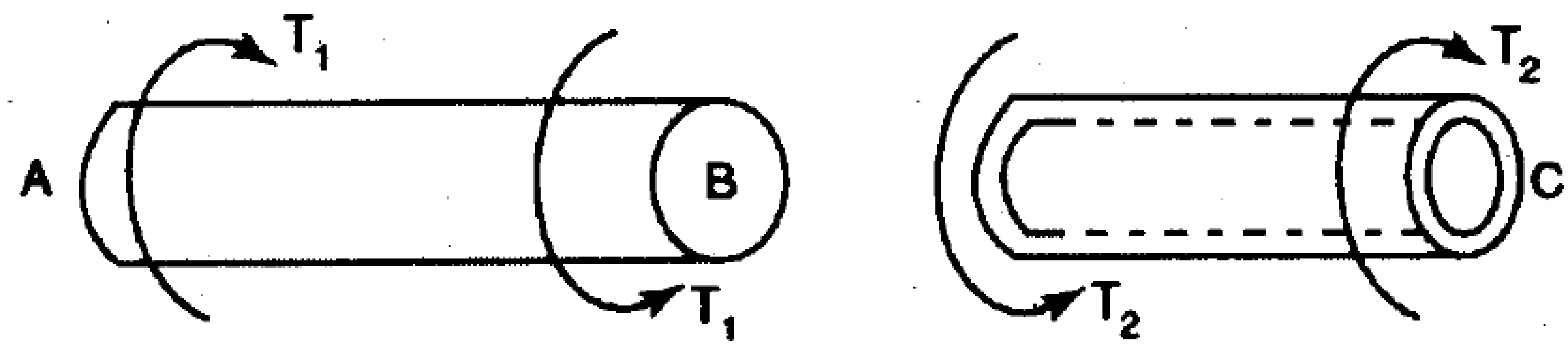


Fig 7.6(b)

Rotation of shaft at *B* in portion *AB,*

$$\theta_1 = \frac{T_1 L_1}{G J_1} = \frac{T_1 \times 600}{G \times \frac{\pi}{32} \times 60^4} = \frac{32 \times 600}{\pi \times 60^4} \frac{T_1}{G}$$

Rotation of shaft at B in portion BC

$$\theta_2 = \frac{T_2 L_2}{G J_2} = \frac{T_2 \times 400}{G \times \frac{\pi}{32} \times (60^4 - 30^4)}$$

$$= \frac{32 \times 400}{\pi \times 60^4 (1 - 0.5^4)} \frac{T_2}{G}$$

Since, $\theta_1 = \theta_2$, for consistency of deformation

$$\frac{600}{60^4} T_1 = \frac{400}{60^4(1-0.5^4)} T_2$$

$$T_1 = 0.7111 T_2$$

But $\quad T_1 + T_2 = T = 2\times10^6$ Nmm

i.e., $\quad (0.7111+1.0)\ T_2 = 2\times10^6$

$\therefore \quad T_2 = 1.1688 \times 10^6$ N-mm

and $\quad T_1 = 0.83117 \times 10^6$ N-mm

Maximum stress in portion *AB*

$$\frac{T_1}{J_1} = \frac{q_{s1}}{R_1}$$

$$q_{s1} = \frac{0.83117 \times 10^6 \times 30}{\frac{\pi}{32} \times 60^4} = 19.60\ N/mm^2$$

Maximum stress in portion *BC*,

$$q_{s2} = \frac{T_2 R_2}{J_2}$$

$$= \frac{1.1688 \times 10^6 \times 30}{\frac{\pi}{32}(60^4 - 30^4)}$$

$$= 29.396\ N/mm^2$$

- Ex. 7.5: *A gun metal sleeve is fixed securely to a steel shaft (Fig.7·7) and the compound shaft is subjected to a torque. If the torque on the sleeve is twice the torque on the shaft find the ratio of the external diameter of sleeve to diameter of the shaft*

Given: $G_{steel} = 2{\cdot}5\ G_{gun\ metal}$

Solution: Let, D = External diameter of the sleeve,

d = Diameter of the shaft,

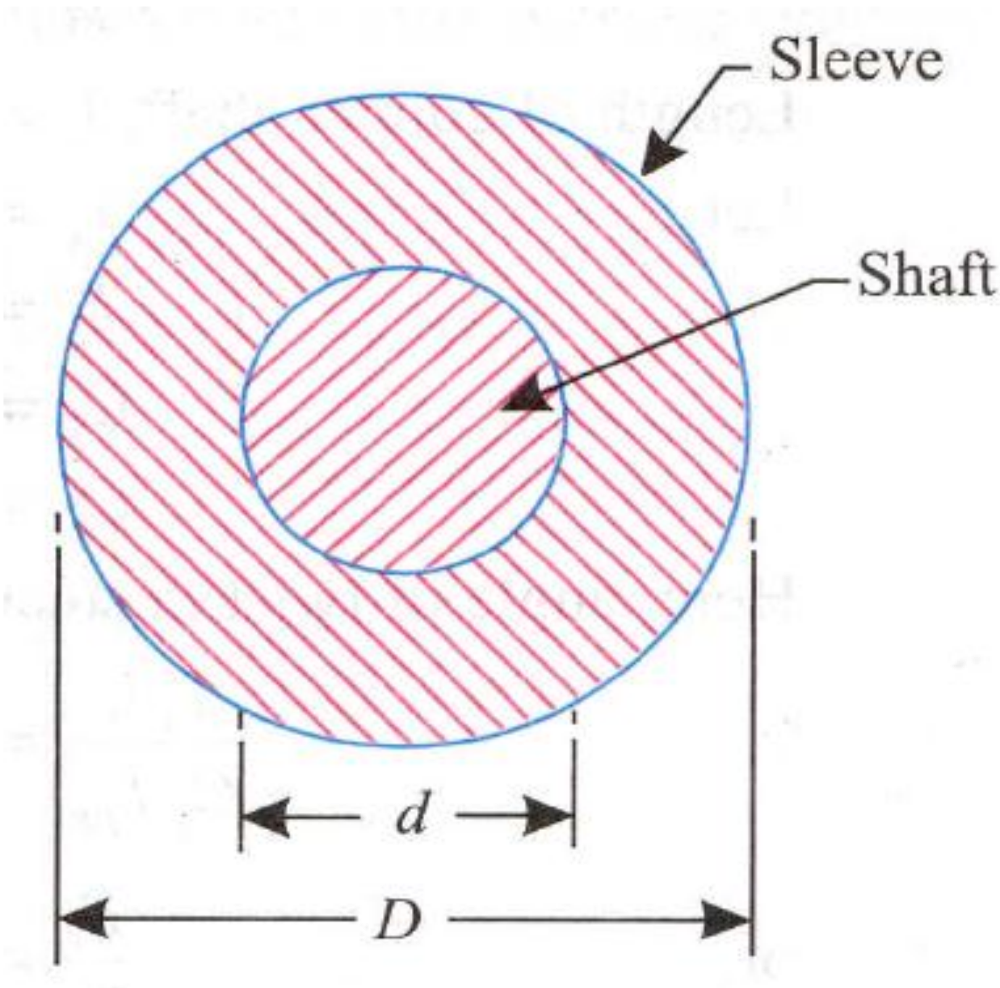


Fig 7.7

T_s= Torque on steel shaft,

T_g= Torque on gun metal sleeve,

= 2 T_s

(Given) G_s= Modulus of rigidity for steel,

G_g= Modulus of rigidity for gun metal,

J_s = Polar moment of inertia for steel shaft, and

J_g= Polar moment of inertia for gun metal sleeve

- **Ratio $\frac{D}{d}$:**

Since the shaft and sleeve are securely fixed together their lengths and the angle of twist at the common surface shall be same *i.e.* l *and* θ *for both are the same.*

From the relation, $\frac{T}{J} = \frac{G\theta}{l}$, we have

$$\frac{\theta}{l} = \frac{T}{GJ}$$

$$\frac{T_s}{G_s J_s} = \frac{T_g}{G_g J_g}$$

$$\frac{T_s}{T_g} = \frac{G_s J_s}{G_g J_g}$$

$$T_g = 2T_s$$

$$G_s = 2.5G_g \ (\boldsymbol{given})$$

$$\therefore \qquad \frac{T_s}{2T_s} = 2.5 \times \frac{J_s}{J_g}$$

$$= 2.5 \times \frac{\frac{\pi}{32} \times d^4}{\frac{\pi}{32}(D^4 - d^4)}$$

$$or, \qquad \frac{D^4 - d^4}{d^4} = 5$$

$$or, \qquad D^4 = 6d^4$$

$$or, \qquad \frac{D}{d} = (6)^{\frac{1}{4}} = 1.565$$

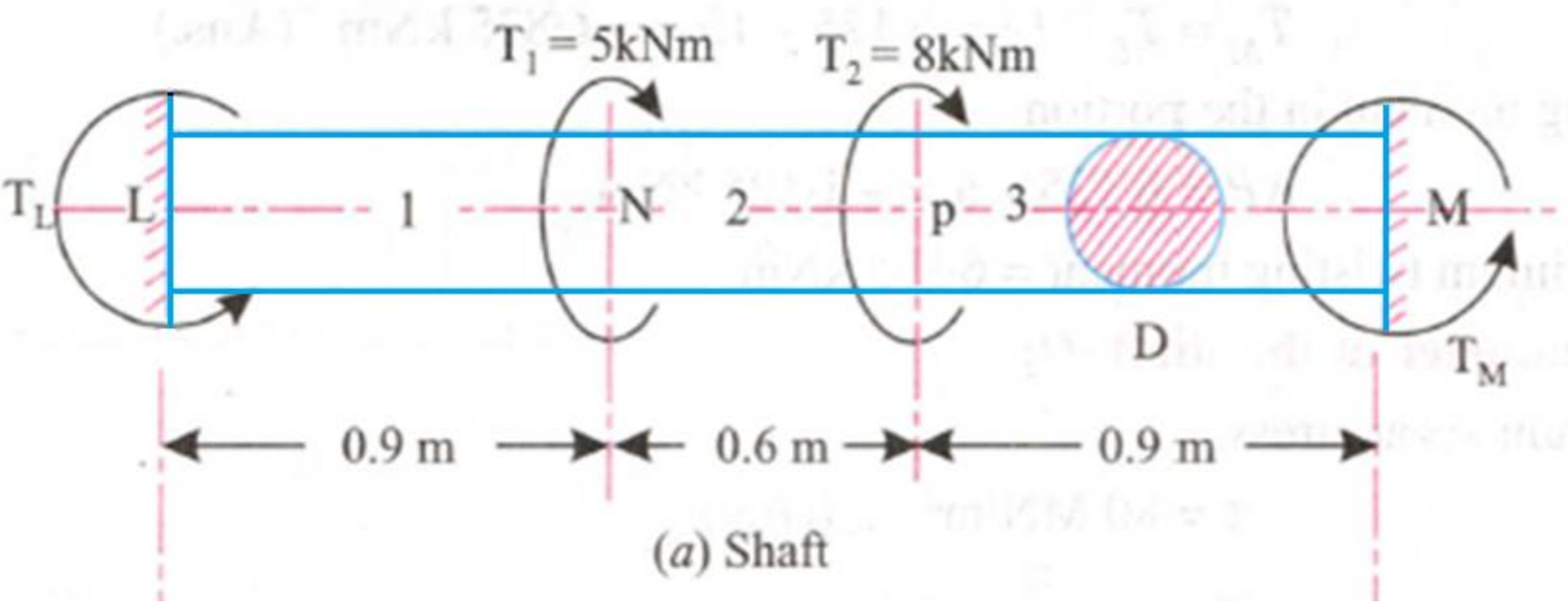
$$hence, \qquad \frac{D}{d} = 1.565\ (\boldsymbol{ans})$$

Ex. 7.6. *Figure 7.8 shows a horizontal shaft LM subjected to axial twisting moments.* ***Determine:***

(i) The end fixing couples in magnitude and direction

(ii) The diameter of the shaft if the maximum shearing stress is not to exceed 80 MN/m^2

(iii) The position of section where the shaft suffers no angular twist



(a) Shaft

Fig 7.8

Solution: Let, T_L= End fixing couple at *L*, and

T_M = End fixing couple at M.

***(i)* End fixing couples :**

Torque on the portion *NP* = $T_L - 5$ kNm

Torque on the portion *PM* = $T_L - 5 - 8$

= $T_L - 13$ kNm

We know, Total angle of twist between Land *M* = $\theta_1 + \theta_2 + \theta_3 = 0$ $(\therefore both\ ends\ are\ fixed)$

$$\therefore \quad \frac{T_L l_1}{GJ} + \frac{(T_L - 5)l_2}{GJ} + \frac{(T_L - 13)l_3}{GJ} = 0$$

$$or, \quad 0.9T_L + 0.6(T_L - 5) + 0.9(T_L - 13) = 0$$

$$or, \quad 0.9T_L + 0.6T_L - 3 + 0.9T_L - 11.7 = 0$$

$$or, \quad 2.4T_L = 14.7$$

$$or, \quad T_L = 6.125\ kNm \quad (\boldsymbol{ans})$$

$$and,$$

$$T_M = T_L - 13 = 6.125 - 13 = -6.875kNm\ (\boldsymbol{ans})$$

Twisting moment in the portion

$$NP = 6.125 - 5 = 1.125 \text{ kNm}$$

∴ Maximum twisting moment = 6.875 kNm

***(ii)* Diameter of the shaft, *D*:**

Maximum shear stress,

$$q = 80N/mm^2........(given)$$

$$T = q \times \frac{\pi}{16} \times D^3$$

$$D^3 = \frac{16T}{\pi q} = \frac{16 \times 6.875 \times 10^6}{\pi \times 80} = 437.67 \times 10^3$$

$$D = 75.925mm \quad (\boldsymbol{ans})$$

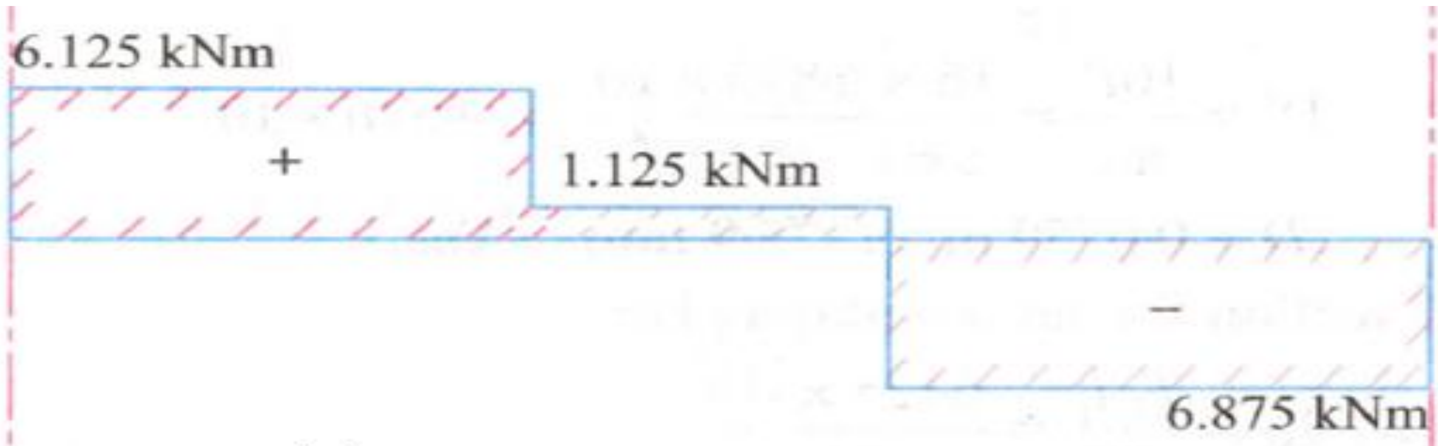
(iii) Position of section for no angular twist:

- Angular twist at N,$\theta_N = \frac{T_L l_1}{GJ} = \frac{6.125\times0.9}{GJ}$

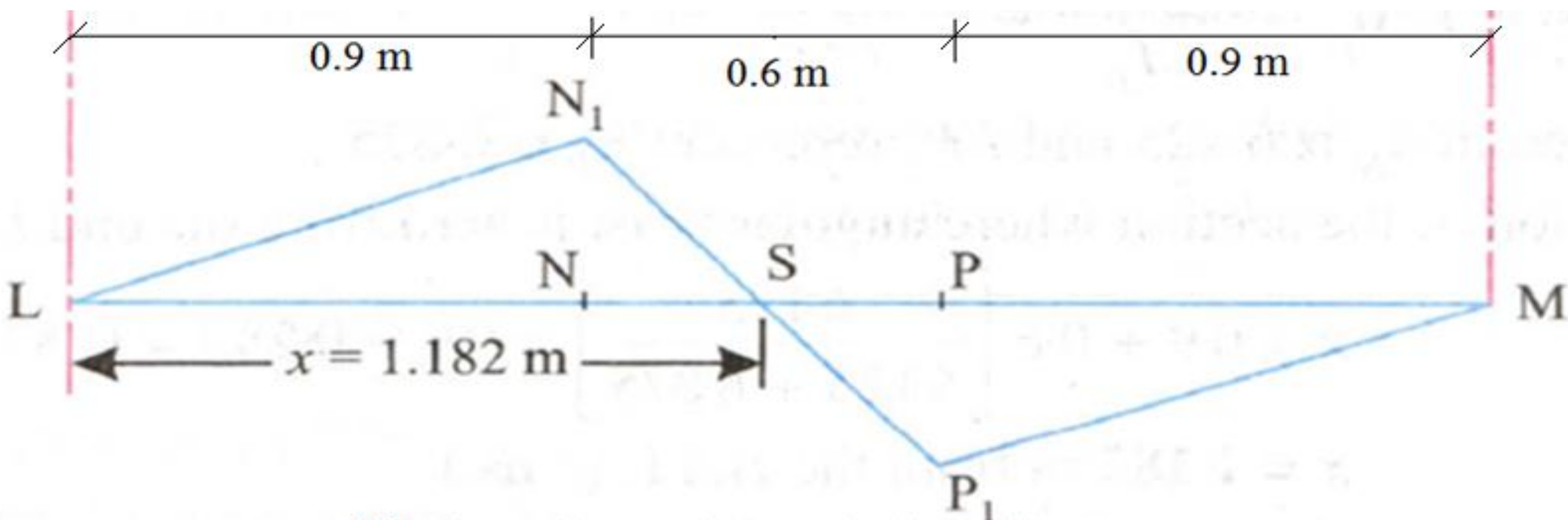
- Angle of twist at *P*, $\theta_p = \frac{T_M \times l_3}{GJ} = \frac{6.875\times0.9}{GJ}$

Let, NN_1 represents $\theta_N \,\alpha$ 6·125 and

PP_1 represents $\theta_P \,\alpha$ 6·875



(a) Torque distribution diagram



(b) Angular twist variation diagram

Fig 7.9

∴ The position of the section where angular twist is zero from the end L,

$$x = 0.9 + 0.6\left[\frac{6.125}{6.125 + 6.875}\right] = 0.9 + 0.282$$

$hence,$

$$x = 1.182m \; from \; the \; end \; L \quad (Ans)$$

Springs

Definition:

A spring may be defined as an elastic member whose primary function is to deflect or distort under the action of applied load; it recovers its original shape when the load is released.

or

Springs are energy absorbing units whose function is to store energy and to restore it slowly or rapidly depending on the particular application.

Springs…

Springs are used in railway carriages, motor cars, scooters, motorcycles, rickshaws, governors etc.

According to their uses, the springs perform the following **functions:**

(i) To absorb shock or impact loading as in *carriage springs.*

(ii) To store energy as in *clock springs.*

Springs…

(iii) To apply forces to and to control motions as in *brakes and clutches.*

(iv) To measure forces as in *spring balances.*

(v) To change the variations characteristic of a member as in *flexible mounting of motors.*

Types of Springs

Helical spring: They are made of wire coiled into a helical form, the load being applied along the axis of the helix.

In these type of springs the major stress is torsional shear stress due to twisting.

Types: (a) Close-coiled, and

(b) Open-coiled

(ii) Spiral springs: They are made of flat strip of metal wound in the form of spiral and loaded in torsion.

In this the major stresses are tensile and compressive due to bending.

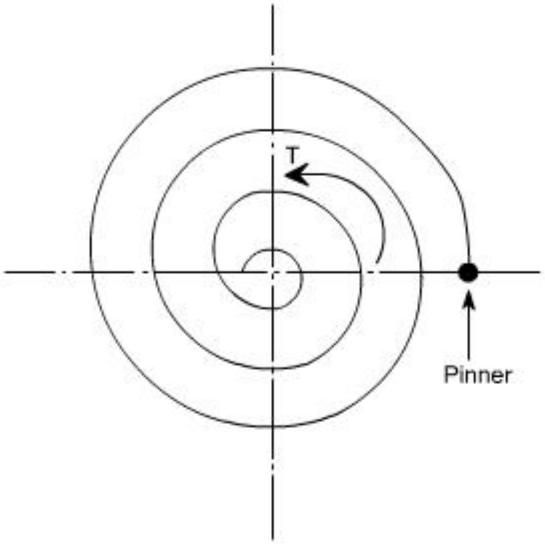
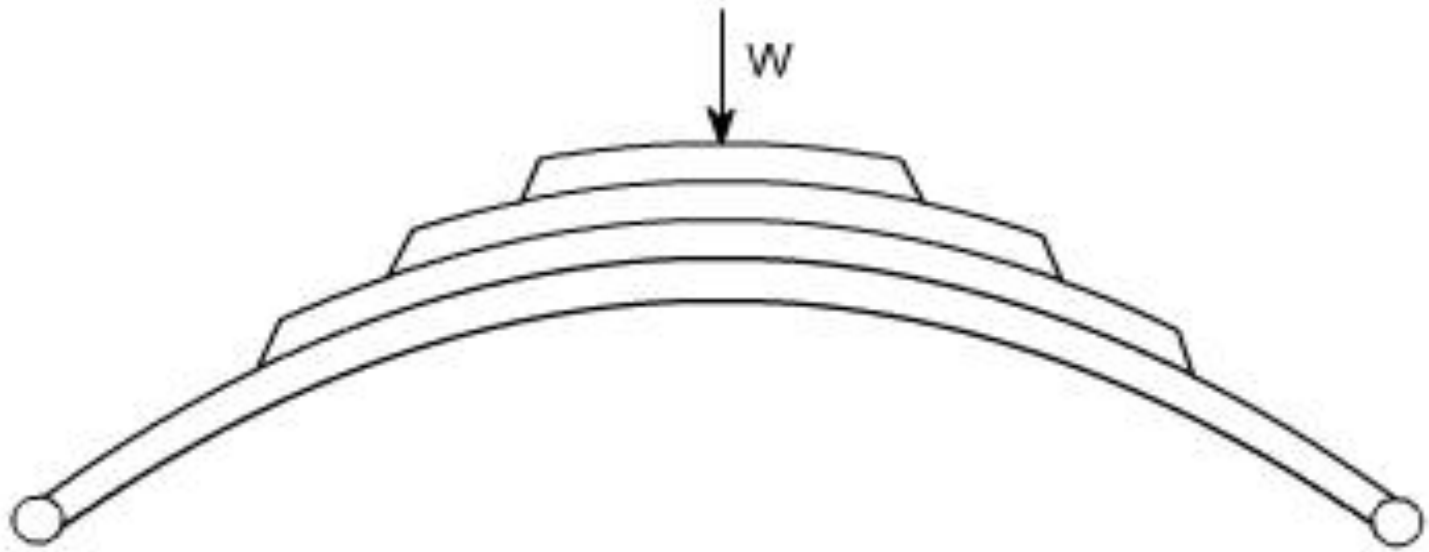


Fig 7.11

(iii) Leaf springs:



These type of springs are used in the automobile suspension system

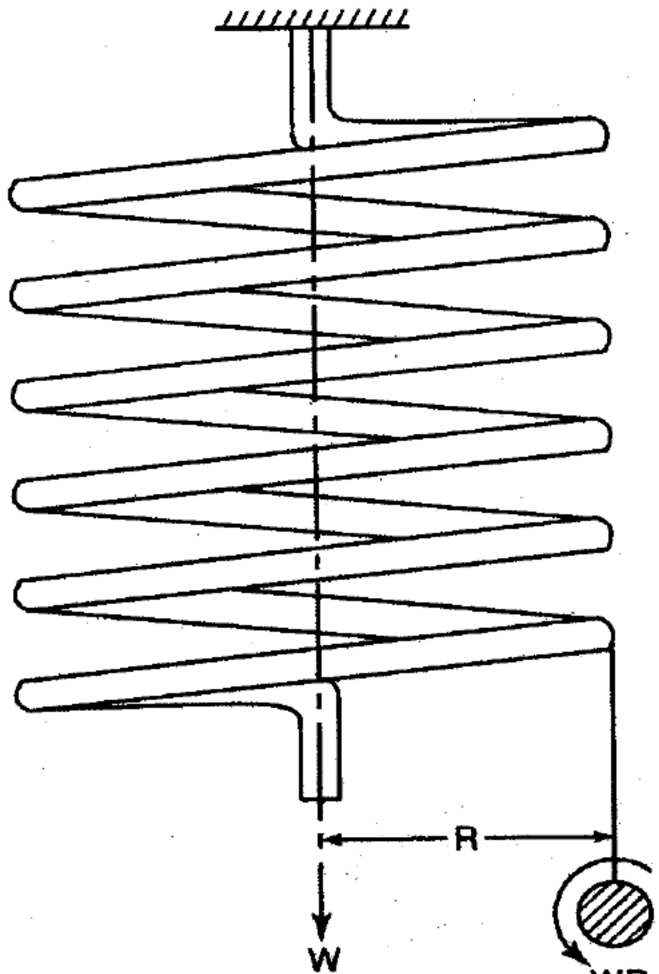
Leaf springs…

They are composed of flat bars of varying lengths clamped together so as to obtain greater efficiency .

Leaf springs may be full elliptic, semi elliptic or cantilever types,

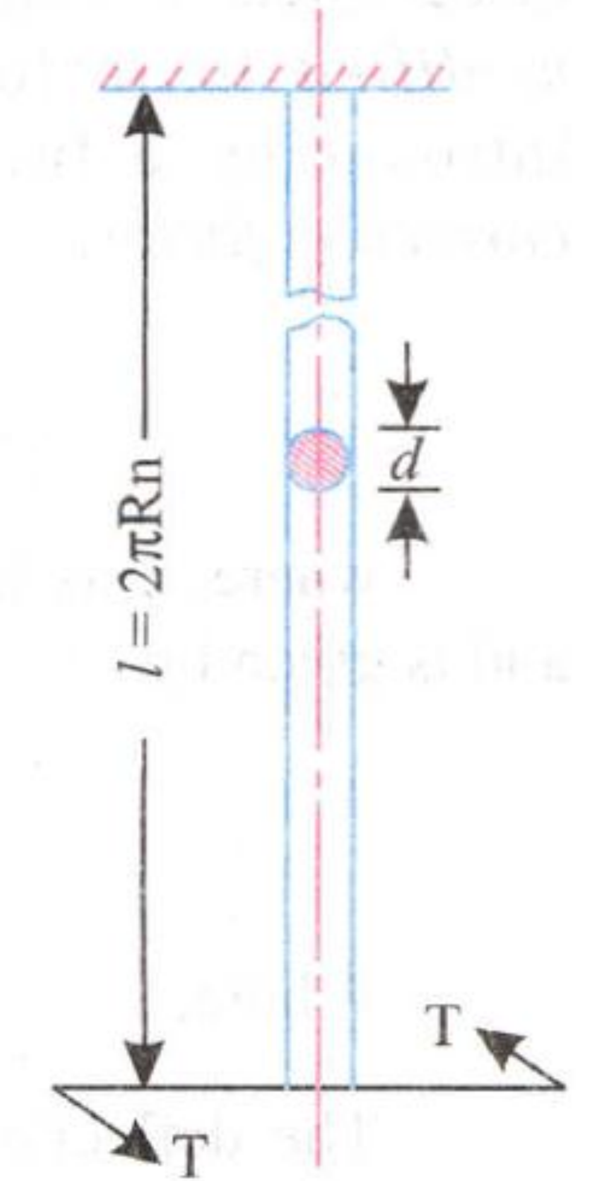
In these type of springs the major stresses which come into picture are tensile & compressive.

Close-coiled helical spring with 'Axial load'



Dr.R.Chinnaraja, Associate Professor,
Division of Structural Engineering, Anna
University, Chennai - 600 025.

when the spring is being subjected to an axial load to the wire of the spring gets be twisted like a shaft.



- **Assumptions:**

(1) The Bending & shear effects may be neglected

(2) For the purpose of derivation of formula, the helix angle is considered to be so small that it may be neglected.

Any one coil of a such a spring will be assumed to lie in a plane which is nearly perpendicular to the axis of the spring. This requires that adjoining coils be close together. With this limitation, a section taken perpendicular to the axis the spring rod becomes nearly vertical.

- Hence to maintain equilibrium of a segment of the spring, only a shearing force V = W and Torque T = WR are required at any cross section.

- In the analysis of springs it is customary to assume that the shearing stresses caused by the direct shear force is uniformly distributed and is negligible

- Let

  W = axial load

  R = mean coil radius

  d = diameter of spring wire

  n = number of active coils

  S = spring index = D / d (For circular wires)

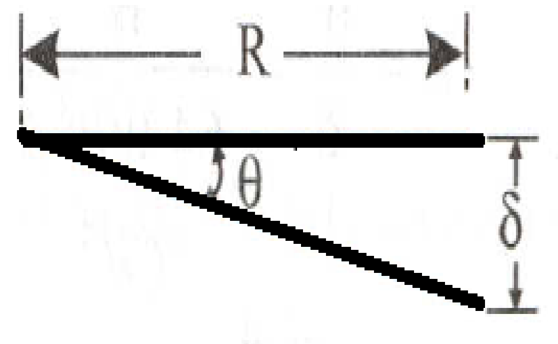
  l = length of spring wire = $2\pi Rn$

  G = modulus of rigidity

  J = Polar moment of inertia

If ‘θ’ is the total angle of twist along the wire and ‘δ’ is the deflection of spring under the action of load W along the axis of the coil, so that

$$\delta = R\,\theta$$



Shear stress, q :

From torsion equation,

$$\frac{T}{J} = \frac{G\theta}{l} = \frac{q}{r}$$

$$\frac{T}{J} = \frac{q}{r}$$

$$T = \frac{qJ}{r} = \frac{q \times \pi d^4}{32} \times \frac{2}{d} = q\frac{\pi}{16}d^3$$

$$or, \qquad q = \frac{16T}{\pi d^3}$$

$$or, q = \frac{16WR}{\pi d^3} \quad --(1)\ (\because T = WR)$$

Again,

$$\frac{T}{J} = \frac{G\theta}{l}$$

$$\theta = \frac{Tl}{GJ} = \frac{WR \times 2\pi Rn \times 32}{G \times \pi d^4}$$

$$= \frac{64WR^2n}{Gd^4}$$

$$\delta = R \times \theta$$

$$\delta = \frac{64WR^3n}{Gd^4} \text{ -----(2)}$$

- ***Wahl' s correction factor:***
- While deriving equations for shear stress (1) and deflection (2) the effect of curvature of spring and direct shear is neglected.

- Eqn. (1) is *modified* to include these effects by introducing a factor *K* called *Wahl's correction factor.*

$$\therefore \quad \boldsymbol{q} = \frac{16WR}{\pi d^3} K$$

- where, *K* is found from *experiments* and is given by

$$K = \frac{4S - 1}{4S - 4} + \frac{0.615}{S}$$

- Where S = spring index = D/d

D = mean coil diameter
d = diameter of spring wire

Spring stiffness: The stiffness is defined as the load per unit deflection

Stiffness of the spring,k:

$$k = \frac{W}{\delta} = \frac{W}{\frac{64WR^3n}{Gd^4}} = \frac{Gd^4}{64nR^3}$$

$$k = \frac{Gd^4}{64R^3n}$$

- **Strain Energy** : The strain energy is defined as the energy which is stored within a material when the work has been done on the material.

Energy stored,U $= \frac{1}{2} \times T \times \theta$

$$= \frac{1}{2} W.R \times \frac{64WR^2n}{Gd^4}$$

$$= \frac{1}{2} \times \frac{1}{2} \times \frac{16WR}{Gd^3} \times \frac{8WR^2n}{d}$$

$$= \frac{1}{4G} \times \frac{16WR}{\pi d^3} \times \frac{16WR}{\pi d^3}\left[2\pi Rn \times \frac{\pi d^2}{4}\right]$$

$$= \frac{1}{4G} . q^2 \times volume\ of\ wire$$

$$i.e, \qquad U = \frac{q^2}{4G} . \times volume\ of\ wire$$

Again, energy stored,

$$U = \frac{1}{2} . T . \theta = \frac{1}{2} . W . R \frac{\delta}{R} = \frac{1}{2} . W . \delta \ \ (\because \delta = R\theta)$$

$$U = \frac{1}{2} W\delta$$

Ex. 7.7. *A closely coiled helical spring is to carry a load of 500 N. Its mean coil diameter(D) is to be 10 times that of the wire diameter(d). Calculate these diameters if the maximum shear stress in the material of the spring is to be 80 N/mm².*

Solution. Load to be carried, $W = 500$ N

Mean coil diameter, $D = 10\,d$,

Shear stress, $q = 80\ N/mm^2$

Diameter,D and d:

Using the relation: $q = \dfrac{16WR}{\pi d^3}, we\ have$

$$80 = \frac{16 \times 500 \times 5d}{\pi d^3}$$

$$\therefore \qquad d^2 = \frac{16 \times 500 \times 5}{\pi 80} = 159.155$$

$$\therefore \qquad d = 12.62\, mm \ \ (Ans)$$

$$and \qquad D = 10d = 10 \times 12.62 = 1262\, mm$$

Ex. 7·8. *A helical spring is made of 12 mm diameter steel wire wound on a 120 mm diameter mandrel. If there are 8 active coils, what is spring constant? Take:* G = 82 GN/m². *What force must be applied to the spring to elongate it by* 40 *mm?*

- **Solution.** Diameter of steel wire,
- d = 12 mm
- Diameter of mandrel, D = 120 mm
- Number of active coils, n = 8
- Modulus of rigidity, G = 82 GN/m^2
- Elongation of the spring, δ= 40 mm

Spring constant:

We know that,

Spring constant = stiffness of spring *(k)*,

$$k = \frac{W}{\delta} = \frac{Gd^4}{64R^3n}$$

$$= \frac{82 \times 10^3 \times (12)^4}{64\left[\frac{120}{2}\right]^3 \times 8}$$

$$k = 15.375 \text{ N/mm}(\boldsymbol{Ans})$$

Force to be applied to the spring, W:

Again, $\frac{W}{\delta} = 15.375 \ (or) \frac{W}{40} = 15.375$

$$W = 615 \ N \quad (\boldsymbol{Ans})$$

Ex:56 *A close-coiled helical spring is to have a stiffness of 900 N/m in compression, with a maximum load of 45N and a maximum shearing stress of 120 N/mm^2. The solid length of the spring (i.e., coils touching) is* 45 *mm. Find:*

(i) The wire diameter,

(ii) The mean coil radius, and

(iii) The number of coils.

Take modulus of rigidity of material of the spring

$= 0{\cdot}4 \times 10^5$ N/mm^2.

Solution:

<u>*given:*</u> $k = 900\ N/m = 0.9N/mm$

$W\ = 45\ N\ :\ \boldsymbol{q} = 120\ N/mm^2;$

$G = 0.4 \times 10^5 N/mm^2$

(i) The wire diameter, d:

$$\delta = \frac{64WR^3n}{Gd^4}$$

$$or, \qquad k = \frac{W}{\delta} = \frac{Gd^4}{64R^3n}$$

$$or, \qquad 0.9 = \frac{0.4 \times 10^5 \times d^4}{64R^3n}$$

$$or, \qquad d^4 = \left(\frac{0.9 \times 64}{0.4 \times 10^5}\right) R^3 n \quad \ldots . (1)$$

Also, $q = \frac{16WR}{\pi d^3}$

$$\therefore \quad 120 = \frac{16 \times 45 \times R}{\pi d^3} \quad or \quad R = \frac{120 \times \pi d^3}{16 \times 45}$$

$$or, \qquad R = 0.5236\, d^3 \ldots \ldots (2)$$

Solid length of the spring when the coils are touching = nd = 45mm

$$n = \frac{45}{d} \ldots \ldots (3)$$

Substituting the values of R and n in eqn.(1), we get

$$d^4 = \left(\frac{0.9 \times 64}{0.4 \times 10^5}\right) \times (0.5236d^3)^3 \times \frac{45}{d}$$

$$= \frac{0.9 \times 64}{0.4 \times 10^5} \times (0.5236)^3 \times 45 \times d^8$$

$$d^4 = \frac{0.4 \times 10^5}{0.9 \times 64 \times (0.5236)^3 \times 45} = 107.5$$

$$\therefore \quad d = (107.5)^{\frac{1}{4}} \approx \mathbf{3.22}\boldsymbol{mm} \ \ (\boldsymbol{Ans})$$

(ii) The mean coil radius, R:

$$R = 0.5236d^3 \;\; \dots\dots\dots\dots \left(Eqn.\,(2)\right)$$

$$= 0.5236 \times (3.22)^3 = 17.48\, mm \;\; (Ans)$$

(iii) The number of coils, n:

$$n = \frac{45}{d} \dots\dots\dots \left(Eqn.\,(3)\right)$$

$$= \frac{45}{3.22} = 13.975 \;\; (Ans)$$

Ex:57. *For a close-coiled helical spring subjected to an axial load of 300 N having 12 coils of wire diameter of* 16*mm, and made with coil diameter of 250 mm, find:*

(i) Axial deflection;

(ii) Strain energy stored;

(iii) Maximum torsional shear stress in the wire;

(iv) Maximum shear stress using Wahl's correction factor.

Take: $G = 80\ GN/m^2$

Solution.

Number of coils, n = 12 coils

Wire diameter, d = 16 mm

Coil diameter, D = 250 mm

Modulus of rigidity, G = 80×10^3 N/mm^2

Axial load, W = 300 N

(i) Axial deflection, δ:

$$\delta = \frac{64WR^2n}{Gd^4} = \frac{64 \times 300 \times (125)^3 \times 12}{80 \times 10^3 \times (16)^4}$$

$$= 85.83\ mm$$ **(Ans)**

(ii) Strain energy stored, U:

$$U = \frac{1}{2}W\delta = \frac{1}{2} \times 300 \times 85.83 = 12874\ Nmm$$

(iii) *Maximum torsional shear stress, q:*

$$q\ = \frac{16WR}{\pi d^3} = \frac{16 \times 300 \times 125}{\pi \times (16)^3}$$

$$= 46.63\ N/mm^2$$

$$\boldsymbol{q\ = 46.63\ N/mm^2}$$ **(Ans)**

(iv) Maximum shear stress using WAHL'S correction factor, q :

$$q = \frac{16WR}{\pi d^3} \times K, where\ K = \frac{4S-1}{4S-4} + \frac{0.615}{S}$$

$$but, S(spring\ index) = \frac{D}{d} = \frac{250}{16} = 15.625$$

$$K = \frac{4 \times 15.625 - 1}{4 \times 15.625 - 4} + \frac{0.615}{15.625}$$

$$= 1.0513 + 0.0394 = 1.0907$$

$$q = 46.63 \times 1.0907 N/mm^2$$

$$= 50.85\ N/mm^2$$

$i.e. q = 50.85\ N/mm^2$ **(Ans)**

- **Close – coiled helical spring subjected to axial torque T or axial couple.**

When a twisting couple is applied to the spring parallel to the axis of the spring wire it produces a bending effect on it.

Depending upon the direction of application of the twisting couple or turning moment the spring coils will close or open out.

In both cases the radius of coils changes and bending stresses will be induced.

- The stresses i.e. maximum bending stress may thus be determined from the bending theory.

$$\sigma_{bmax} = \frac{My}{I}$$

$$= \frac{Td/2}{\frac{\pi d^4}{64}}$$

$$\sigma_{bmax} = \frac{32T}{\pi d^3}$$

Let, n_1 = Number of coils before application of twisting moment,

n_2 = Number of coils after application of twisting moment,

θ =Angle of rotation

R_1 = Mean radius of coil,

R_2 = Changed radius of coil,

σ_b = Bending stress, and

E = modulus of elasticity.

Initial curvature $= \dfrac{1}{R_1}$

Final curvature $= \dfrac{1}{R_2}$

Change in curature $= \dfrac{1}{R_2} - \dfrac{1}{R_1}$

Also, as per bending equation,

$$\frac{M}{I} = \frac{E}{R} \ or \ \frac{1}{R} = \frac{M}{EI}$$

$$\frac{M}{EI} = \frac{1}{R_2} - \frac{1}{R_1}$$

Since length of the wire remains unchanged before and after applying the twisting couple,

$$l = 2\pi R_1 n_1 = 2\pi R_2 n_2$$

$$but, \quad \theta = \text{Final helix angle} - \text{initial helix angle}$$

$$= (2\pi n_2 - 2\pi n_1)$$

$$\frac{M}{EI} = \frac{1}{R_2} - \frac{1}{R_1} = \frac{1}{\frac{l}{2\pi n_2}} - \frac{1}{\frac{l}{2\pi n_1}} = \frac{2\pi}{l}(n_2 - n_1)$$

$$= \frac{\theta}{l}$$

$$\theta = \frac{Ml}{EI}$$

$$= \frac{M \times 2\pi R n}{E \times \frac{\pi}{64} d^4} = \frac{128 M R n}{E d^4}$$

Also, $\sigma_b = \frac{M}{Z} = \frac{My}{I}$

$$= \frac{Md/2}{\frac{\pi d^4}{64}}$$

$$= \frac{32M}{\pi d^3}$$

Energy stored, $U = \frac{1}{2} M\theta = \frac{1}{2} M.\frac{Ml}{EI} = \frac{1}{2}\frac{M^2 l}{EI}$

$$= \frac{1}{2}.\frac{\sigma_b.\pi d^3}{32} \times \frac{\sigma_b \pi d^3}{32} \times \frac{l \times 64}{E \times \pi d^4}$$

$$= \frac{\pi^2 d^6\ \sigma_b{}^2 l \times 64}{2 \times 32 \times 32 \times 6 \times E \times \pi d^4} = \frac{\sigma_b{}^2 \times \pi d^2 l}{8E \times 4}$$

$$= \frac{\sigma_b{}^2}{8E} \times volume\ of\ spring\ wire$$

- **Ex:58.** *A closely coiled helical spring made of wire 5 mm in diameter and having an inside diameter of 40 mm joins two shafts. The effective number of coils between the shafts is 15 and 735 Watt is transmitted through the spring at 1000 r.p.m. Calculate the relative axial twist in degrees between the ends of spring and also the intensity of bending stress in the material. E = 200 GN/m*^{2}.

Solution. Diameter of wire, d = 5mm

Mean diameter of coil, D =40 +5 = 45mm

But, $$R = \frac{45}{2} = 22.5mm$$

We know that, $$P = \frac{2\pi NT}{60}$$

$$or, \quad 735 \times 1000 = \frac{2\pi NM}{60} (\because M = T)$$

$$or, \quad M = \frac{735 \times 10^3 \times 60}{2\pi \times 1000} = 7018.73Nmm$$

$$\text{Now,}\theta = \frac{128MRn}{Ed^4} = \frac{128\times7018.73\times22.5\times15}{200\times10^3\times(5)^4}$$

$$or, \theta = 2.43\ \ radians = 139^{\circ}\ \ (\boldsymbol{Ans})$$

$$\sigma_b = \frac{32M}{\pi d^3} = \frac{32\times7018.73}{\pi(5)^3} = 572\ N/mm^2\ \ \textbf{(Ans)}$$

Ex:59. *A close-coiled helical spring made of round steel wire* 6 *mm diameter having* 10 *complete turns* is *subjected to an axial couple M. The mean coil radius is* 42mm. If the *maximum bending stress in spring wire is not to exceed 240 MN/m*^{2}*, determine:*

(i) The magnitude of *axial couple* M;

(ii) The angle through which one end of spring is turned relative to the other end.

Take: $E_{steel} = 200\ GN/m^2$

Solution. Diameter of steel wire,

$d = 6$ mm

Number of complete turns, $n = 10$

Mean coil radius, $R = 42$ mm

Maximum bending stress, $\sigma_b = 240$ N/mm^2

$E_{steel} = 200\ GN/m^2$

(i) Axial couple, M:

Using the relation:

$$\sigma_b = \frac{32M}{\pi d^3} \text{ ,we have } 240 = \frac{32M}{\pi \times (6)^3}$$

$$M = \frac{240 \times \pi \times (6)^3}{32} = 5089\ Nm$$ **(Ans)**

(ii) Angle of rotation, θ*:*

Using the relation: $\theta = \frac{128MRn}{Ed^4}, we\ have$

$$\theta = \frac{128 \times 5089 \times 42 \times 10}{200 \times 10^3 \times (6)^4} \times \frac{180}{\pi}\ degree$$

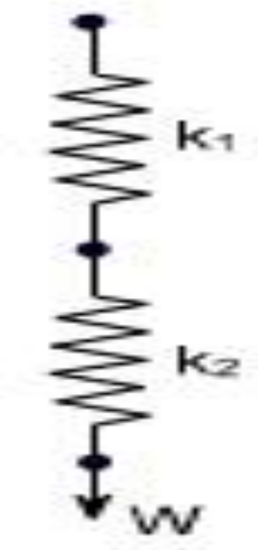
$$= 60.47^\circ$$

$\mathrm{H}ence, \theta = \ 60.47^\circ$ **(Ans)**

Springs in Series:

If two springs of different stiffness are joined and carry a common load W, they are said to be connected in series and the combined stiffness and deflection are given by the following equation.

$$\frac{W}{k} = \delta_1 + \delta_2 = \frac{W}{k_1} + \frac{W}{k_2}$$



$or,$

$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$

Springs in parallel:

If the two springs are joined in such a way that they have a common deflection 'δ'; then they are said to be connected in parallel. In this case the load carried is shared between the two springs and total load $W = W_1 + W_2$

$$\delta = \delta_1 = \delta_2$$

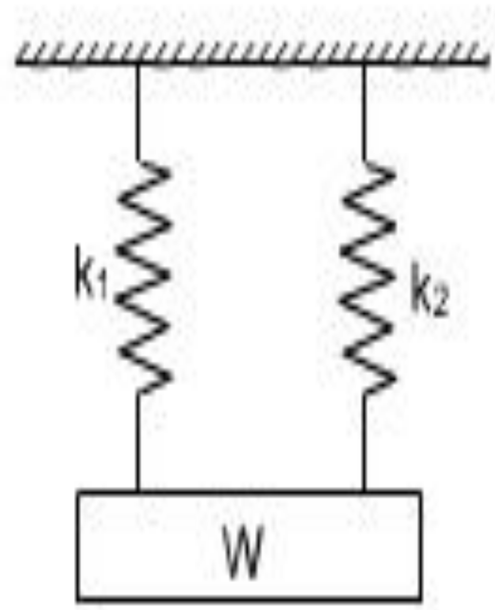
$$\delta = \frac{W}{k} = \frac{W_1}{k_1} = \frac{W_2}{k_2}$$

Thus $$W_1 = \frac{Wk_1}{k}$$

$$W_2 = \frac{Wk_2}{k}$$

Further

$$W = W_1 + W_2$$



$$\delta k = \delta k_1 + \delta k_2$$

$thus$ $$k = k_1 + k_2$$

Ex 7.13. A bumper is to be designed to arrest a wagon weighing 500kN moving at 18km/hours. Sizes of buffer springs available are having diameter = 30mm, mean radius = 100mm, number of turns = 18, modulus of rigidity = 80 kN/mm^2 and maximum compression permitted = 200mm. Find the number of springs required for the buffer.

Solution:

Speed of wagon, v = 18km/h

$$= \frac{18000}{60 \times 60} = 0.5 m/sec$$

$$\therefore Energy\ of\ wagon = \frac{1}{2}mv^2$$

$$= \frac{1}{2} \times \frac{500 \times 10^3}{9.81} \times 0.5^2 = 6371\ Nm$$

Let W be the gradually applied load on spring,

Now, $\delta = \dfrac{64WR^3n}{Gd^4}$

$$200 = \frac{64 \times W \times 100^3 \times 18}{80 \times 10^3 \times 30^4}$$

$$\therefore w = 11250\ N$$

Energy absorbed by each spring

= work done by spring in compression

$$= \frac{1}{2} W\delta = \frac{1}{2} \times 11250 \times 200$$

$$= 1.125 \times 10^6 \ Nmm$$

$$\therefore No. of \ springs \ required = \frac{6371 \times 10^3}{1.125 \times 10^6}$$

= 5.663

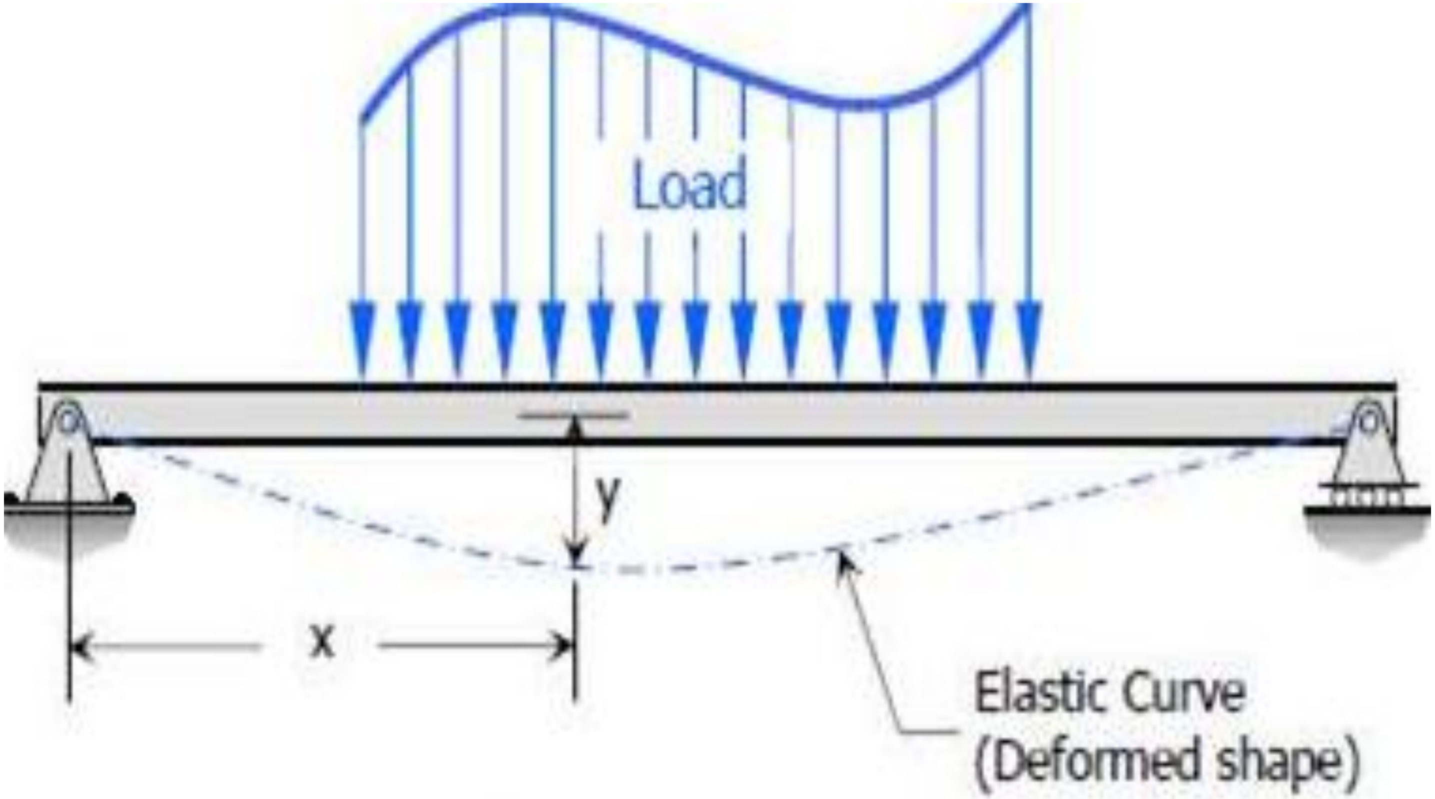
$$\therefore \boldsymbol{Use \ 6 \ springs}$$

UNIT – 4

DEFLECTION OF BEAMS AND COLUMN THEORIES

DEFLECTION OF BEAMS

- The deformation of a beam is usually expressed in terms of its deflection from its original unloaded position.
- The deflection is measured from the original neutral surface of the beam to the neutral surface of the deformed beam.
- The configuration assumed by the deformed neutral surface is known as the elastic curve of the beam.



Elastic curve

- Accurate values for these beam deflections are sought in many practical cases:

- ✓ In buildings, floor beams cannot deflect excessively to avoid the undesirable psychological effect of flexible floors to the occupants and to minimize or prevent distress in brittle-finish materials;

- Elements of machines must be sufficiently rigid to prevent misalignment and to maintain dimensional accuracy under load;

- likewise, information on deformation characteristics of members is essential in the study of vibrations of machines as well as of stationary and flight structures.

Methods of Determining Beam Deflections

- Double integration method
- Macaulay’s method
- Area –moment method (Mohr’s Theorem)
- Conjugate beam method
- Strain energy method
- Method of super position

DIFFERENTIAL EQUATION FOR DEFLECTION

In calculus, the radius of curvature of a curve, y = f(x) is given by

$$R = \frac{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}{|d^2y/dx^2|}$$

In the derivation of flexural formula, the radius of curvature of a beam is given as $R = \frac{EI}{M}$

- Deflection of beams is so small, such that the slope of the elastic curve dy/dx is very small, and squaring this expression the value becomes practically negligible, hence

$$R = \frac{1}{\frac{d^2y}{dx^2}} = \frac{1}{y''}$$

$$y'' = \frac{1}{R} = \frac{M}{EI}$$

- Thus, EI / M = 1 / y''

- If **EI** is constant, the equation may be written as:

$$EIy'' = M$$

where **y** is the deflection of the beam at any distance **x**.

E is the modulus of elasticity of the beam,

I represents the second moment of area about the neutral axis, and

M represents the bending moment at a distance x from the end of the beam.

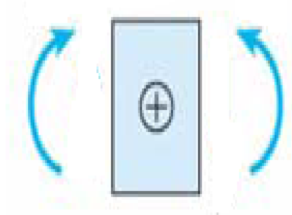
- The first integration **y'** yields the slope of the elastic curve and the second integration **y** gives the deflection of the beam at any distance **x**.
- The resulting solution must contain two constants of integration since **EI y" = M** is of second order.
- These two constants must be evaluated from known conditions concerning the slope deflection at certain points of the beam

Sign convention for Double Integration method & Macaulay's method:

1. For Bending moment:

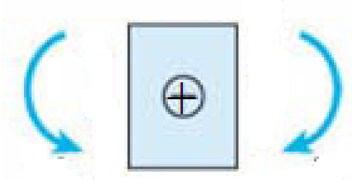
If , $EI\,\frac{d^2y}{dx^2} = -(M_x)$

$then,\ M_x$ is Sagging moment positive



If , $EI\,\frac{d^2y}{dx^2} = M_x$

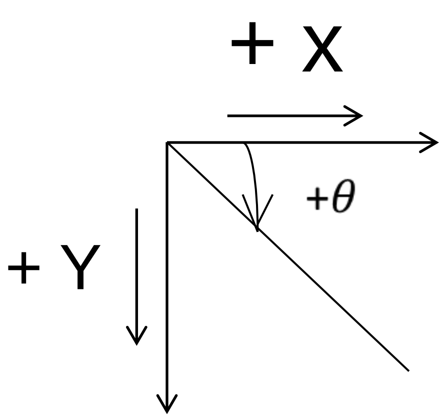
$then,\ M_x$ is Hogging moment positive

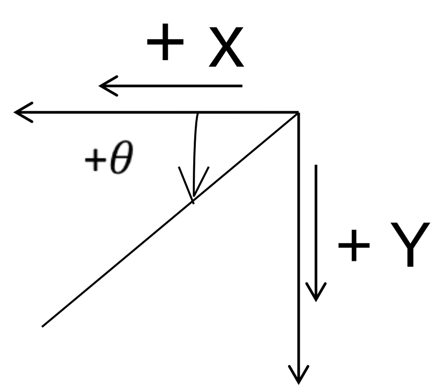


2. For slope and deflection:

(i) if you measure x from left end, then clockwise slope is positive and downward deflection is positive

(ii) if you measure x from right end, then anti-clockwise slope is positive and downward deflection is positive

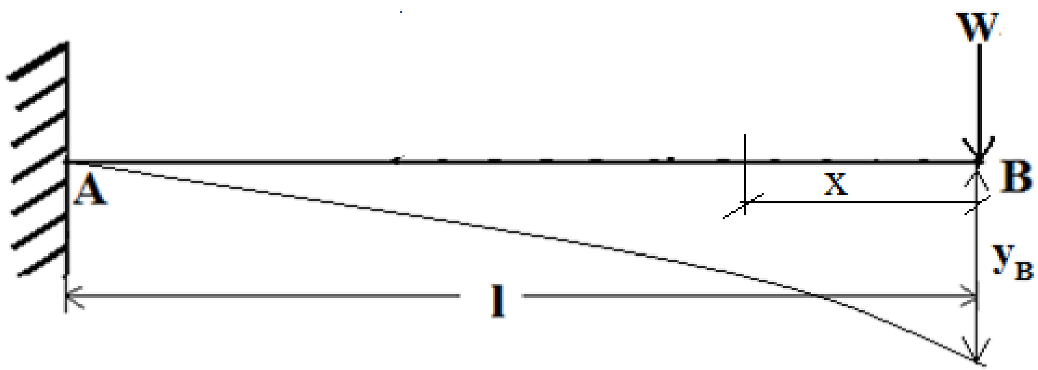




DOUBLE INTEGRATION METHOD

Ex:8.1 A cantilever is subjected to a point load at free end. Derive equations for slope and deflection and determine the slope and deflection at the free end.

Soln:



$$EI\,\frac{d^2y}{dx^2} = -(M_x)$$

$$= -(-Wx)$$

$$= Wx$$

Integrating with respect to x, we get

$$EI\,\frac{dy}{dx} = \frac{wx^2}{2} + C_1 \longrightarrow (1)$$

Integrating again with respect to x, we get

$$EIy = \frac{wx^3}{6} + C_1 x + C_2 \longrightarrow (2)$$

The constants of integration can be determined by applying the known boundary conditions.

i.e., at $x = l, \frac{dy}{dx} = 0 \quad \Rightarrow \quad C_1 = -\frac{Wl^2}{2}$

at $x = l, \mathrm{y} = 0 \quad \Rightarrow \quad 0 = \frac{Wl^3}{6} - \frac{Wl^3}{2} + C_2$

$$\therefore C_2 = \frac{Wl^3}{3}$$

$$\therefore EI\ \frac{dy}{dx} = \frac{Wx^2}{2} - \frac{Wl^2}{2} \longrightarrow \mathrm{I}$$

$$EIy = \frac{Wx^3}{6} - \frac{Wl^2}{2}x + \frac{Wl^3}{3} \longrightarrow \mathrm{II}$$

To find slope at B, put x=o in eqn I,

we get $EI\ \theta_B = -\ \frac{Wl^2}{2}$

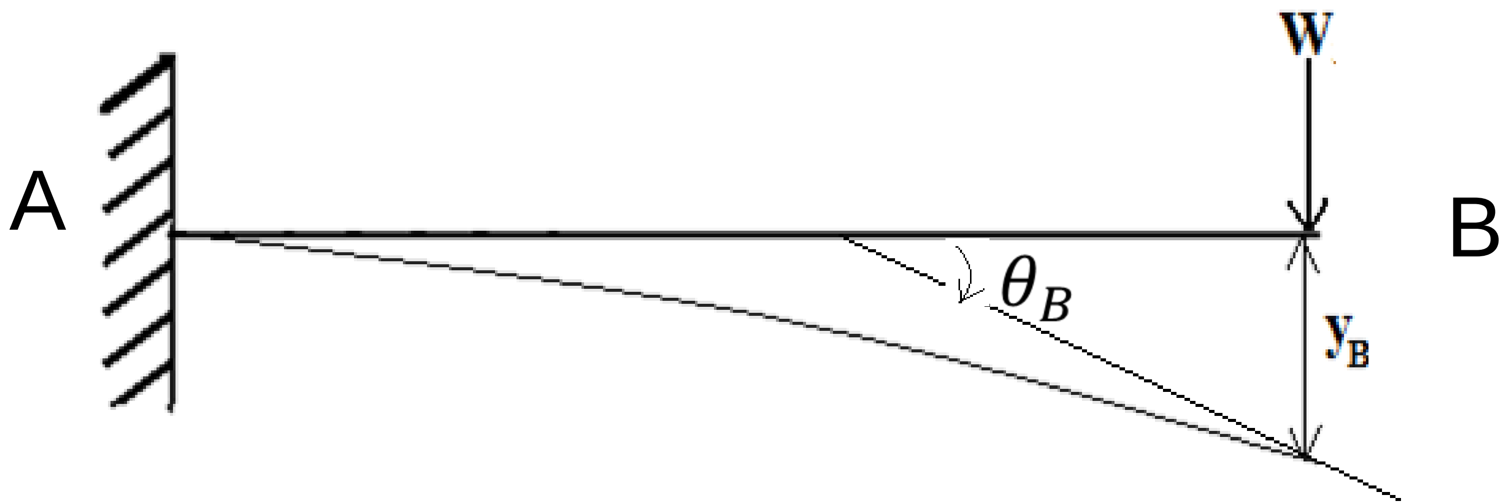
$$\therefore \theta_B = -\ \frac{Wl^2}{2EI}$$

(-ve sign indicates the slope is in clockwise direction since x is measured from right end)

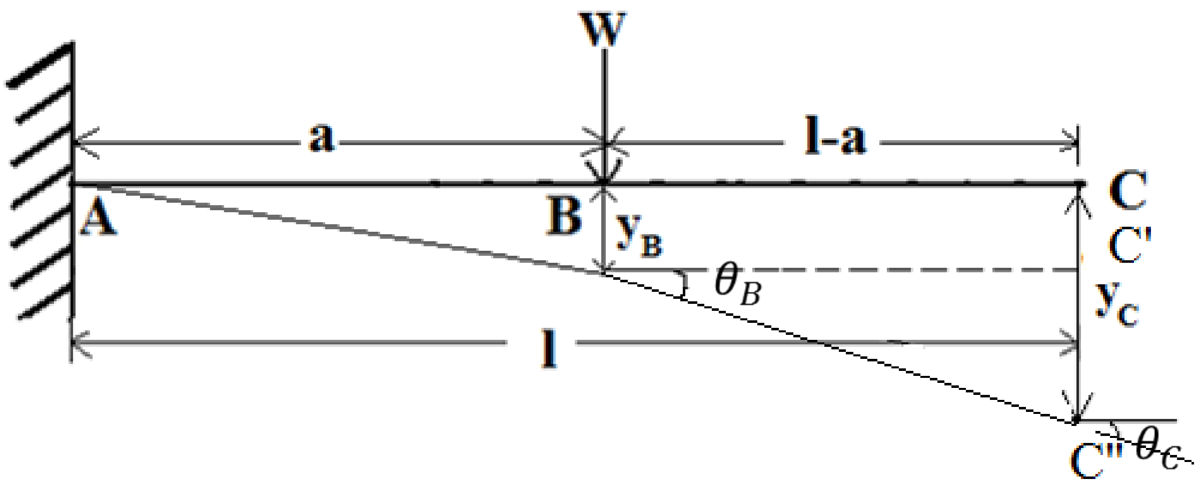
To find deflection at B, put x=0 in eqn II

we get, $EIy_B = \dfrac{Wl^3}{3}$

$$\therefore y_B = \frac{Wl^3}{3EI} \quad \text{(Downward)}$$



Ex:8.2 A Cantilever carries a point load at a distance ‘a’ from the fixed end. Find the slope and deflection at the free end.



Soln:

$$\theta_B = \frac{Wa^2}{2EI}; \quad y_B = \frac{Wa^3}{3EI}$$

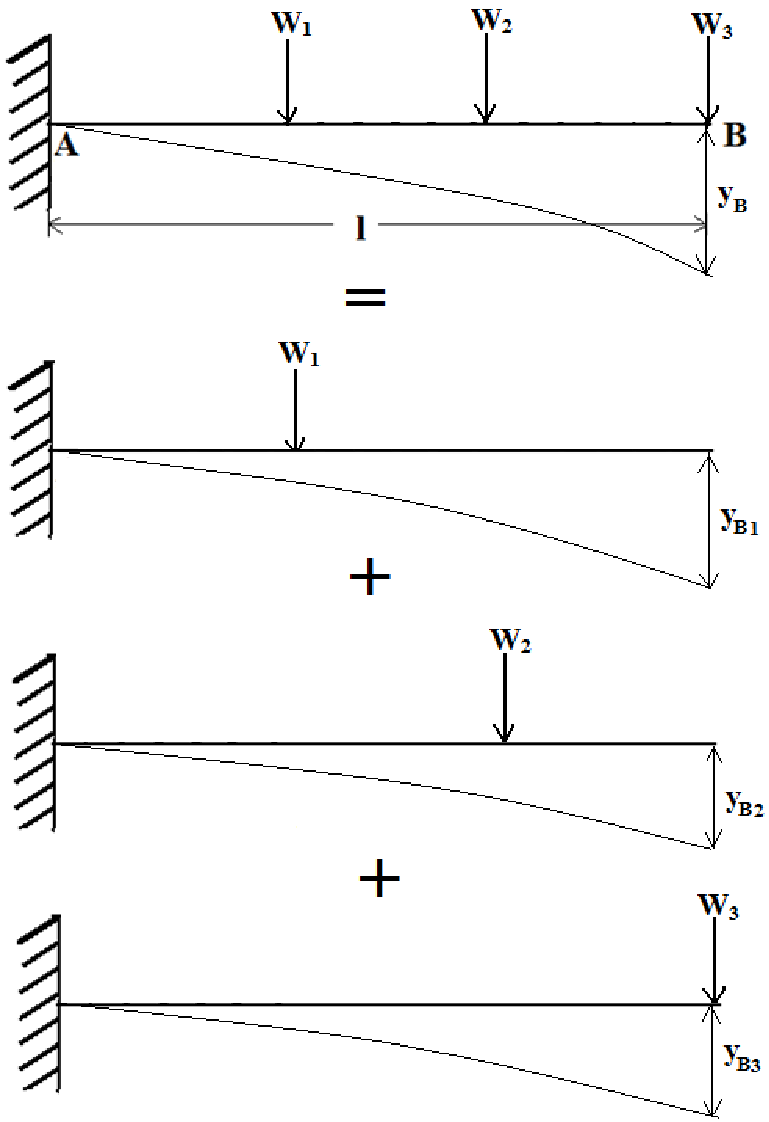
$$\theta_C = \theta_B = \frac{wa^2}{2EI}$$

$$y_c = CC' + C'C“$$

$$= y_B + \theta_B(l - a)$$

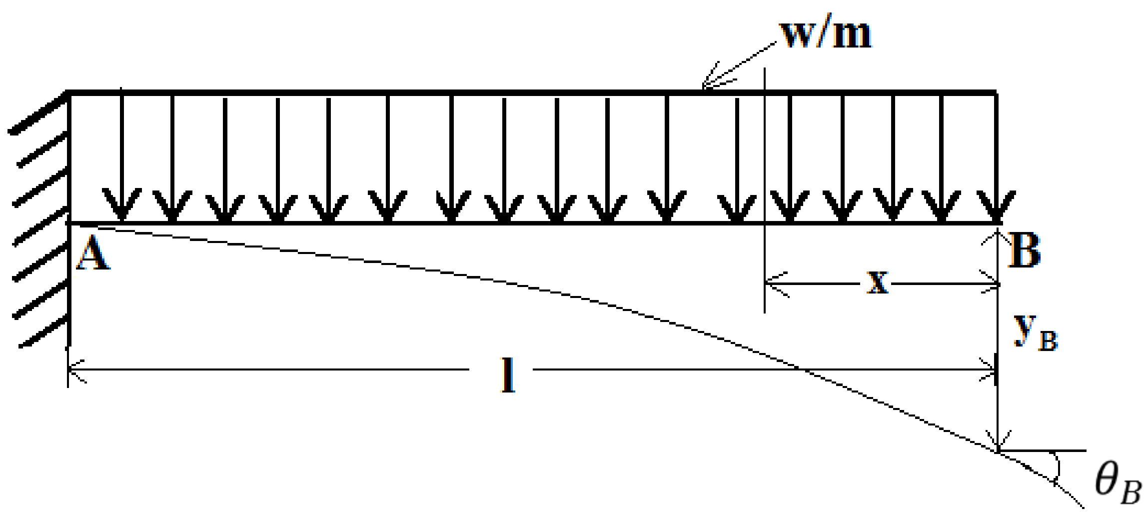
$$= \frac{wa^3}{3EI} + \frac{wa^2}{2EI}(l - a)$$

Ex:63



$$y_B = y_{B1} + y_{B2} + y_{B3}$$

Ex:64 A Cantilever is subjected to u.d.l over the entire length. Derive equations for slope and deflection and find the slope and deflection at free end.



Soln:

$$EI\,\frac{d^2y}{dx^2} = -(M_x) = -\left[-\frac{wx^2}{2}\right] = \frac{wx^2}{2}$$

On integration,

$$EIy' = \frac{wx^3}{6} + C_1 \longrightarrow (1)$$

On integrating once again,

$$EIy = \frac{wx^4}{24} + C_1x + C_2 \longrightarrow (2)$$

At fixed end, (i.e., at x=l); y=0, & y'=0,

Put $x = l, y = 0\ in$ 1, $EI(0) = \frac{wl^3}{6} + C_1$

$$C_1 = -\frac{Wl^3}{6}$$

Put $x = l, y = 0\ in$ 2,

$$0 = \frac{wl^4}{24} + C_1 l + C_2$$

$$0 = \frac{wl^4}{24} - \frac{Wl^3}{6} l + C_2$$

$$0 = \frac{wl^4}{24} - \frac{Wl^4}{6} + C_2$$

$$C_2 = \frac{3wl^4}{24}$$

$$\therefore C_2 = \frac{wl^4}{8}$$

Slope equation:

$$EIy' = \frac{Wx^3}{6} - \frac{Wl^3}{6} \longrightarrow \mathrm{I}$$

Deflection equation:

$$\mathrm{E}Iy = \frac{wx^4}{24} - \frac{Wl^3}{6}x + \frac{wl^4}{8} \longrightarrow \mathrm{II}$$

To find slope at the free end:

Put x=0 in equation I, we get

$$EI\theta_B = -\frac{Wl^3}{6}$$

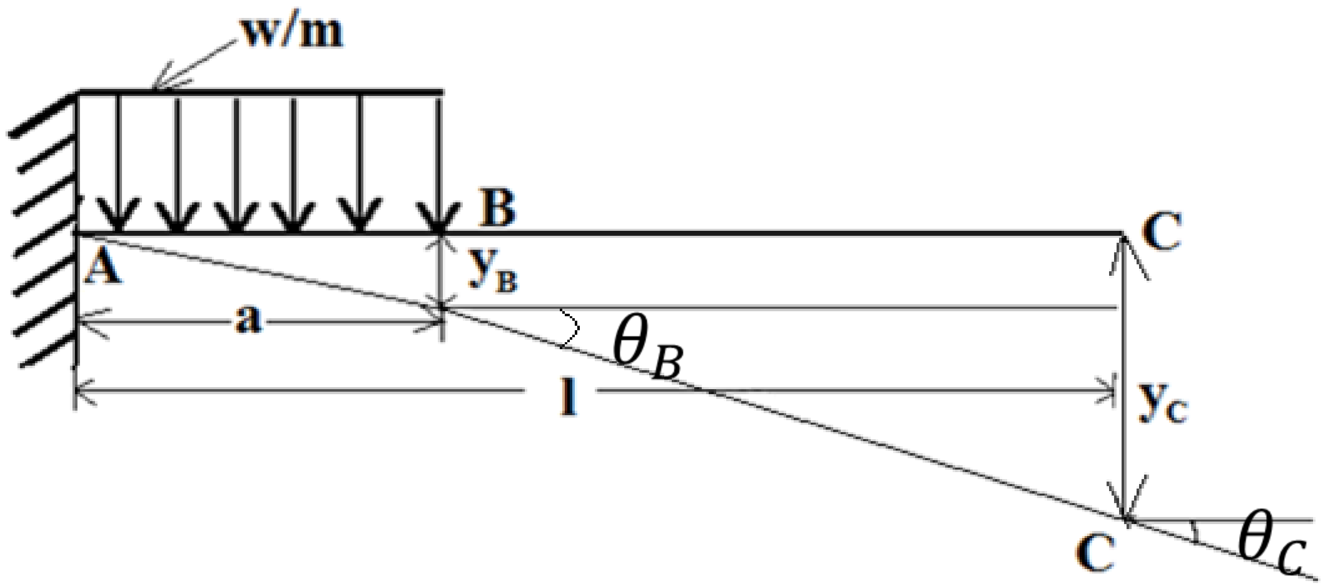
(-ve sign indicates the slope in clockwise)

To find deflection at free end :

Put x=0 in eq II, we get

$$y_B = \frac{wl^4}{8EI}$$

Ex 8.5: A Cantilever carries u.d.l over a span 'a' from the fixed end. Find $\theta_{B,} y_{B,} \theta_{C,} and\ y_C$



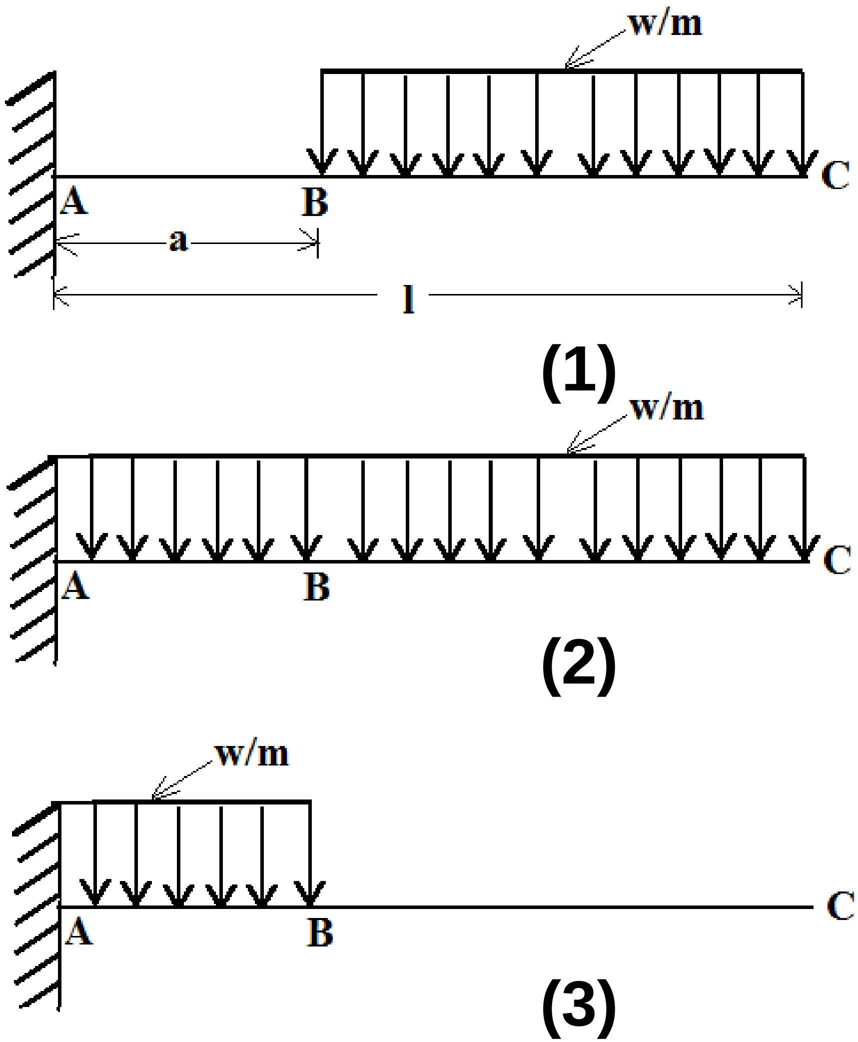
$$\theta_B = \frac{wa^3}{6EI} \; ; \qquad y_B = \frac{wa^4}{8EI}$$

$$\theta_c = \theta_B = \frac{wa^3}{6EI}$$

$$y_C = y_B + \theta_B(\mathrm{l} - \mathrm{a})$$

$$= \frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l - a)$$

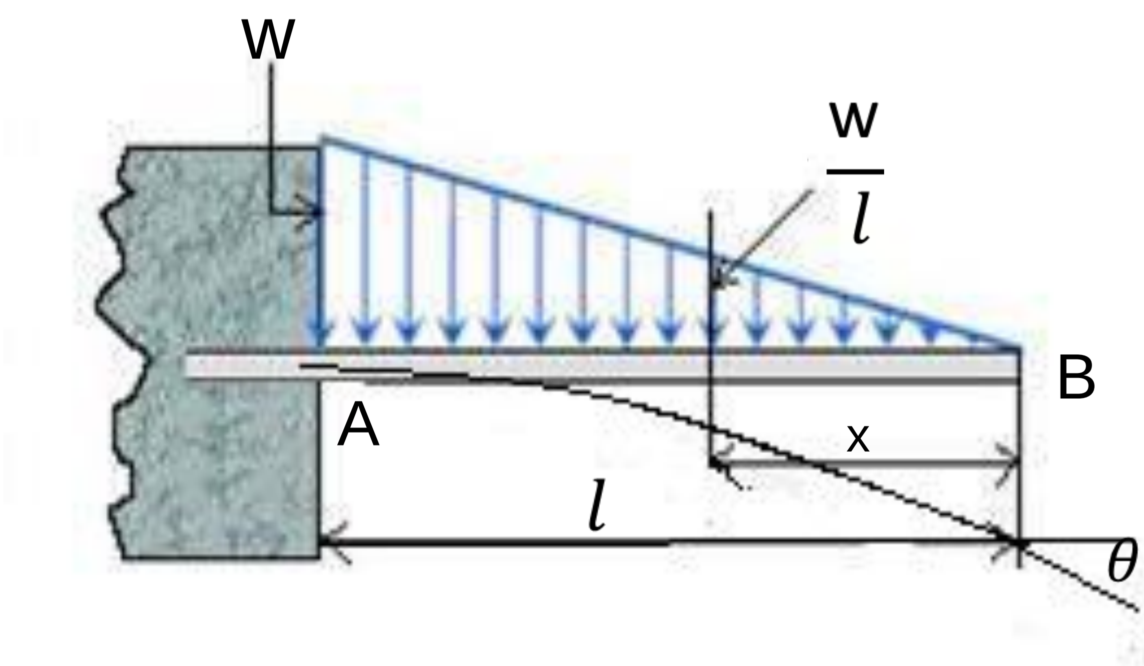
Ex :66. Cantilever with u.d.l over a length from the free end.



$$(1) = (2) - (3)$$

By Principle of superposition

Ex:67. A Cantilever is subjected to triangular load as shown in Fig. Derive equations for slope and deflection and find the slope and deflection at the free end.



$$EI\,\frac{d^2y}{dx^2} = -(M_x)$$

$$= -\left[-\frac{1}{2} \times x \times \frac{wx}{l} \times \frac{x}{3}\right]$$

$$= \frac{wx^3}{6l}$$

$$\therefore EIy' = \frac{wx^4}{24l} + C_1 \longrightarrow \text{(1)}$$

$$and\ EIy = \frac{wx^5}{120\ l} + C_1x + C_2 \longrightarrow \text{(2)}$$

Applying boundary conditions,

i.e., At $x = l, y' = 0 \Rightarrow$

① $\Rightarrow \quad 0 = \frac{wl^3}{24} + C_1$

$$\therefore C_1 = -\frac{wl^3}{24}$$

at $x = l, y = 0$; ② $\Rightarrow \quad 0 = \frac{wl^4}{120} + C_1 l + C_2$

$$0 = \frac{wl^4}{120} - \frac{wl^4}{24} + C_2$$

$$\frac{4wl^4}{120} = C_2 \qquad\qquad \therefore C_2 = \frac{wl^4}{30}$$

$$\therefore EIy' = \frac{wx^4}{24l} - \frac{wl^3}{24} \longrightarrow I$$

$$and\ EIy = \frac{wx^5}{120\,l} - \frac{wl^3}{24}x + \frac{wl^4}{30} \longrightarrow II$$

To find slope at free end put x=0 in equation I

$$EI\theta_B = -\frac{wl^3}{24}$$

$$\theta_B = -\frac{wl^3}{24EI}$$

$$= \frac{wl^3}{24EI} \; clockwise$$

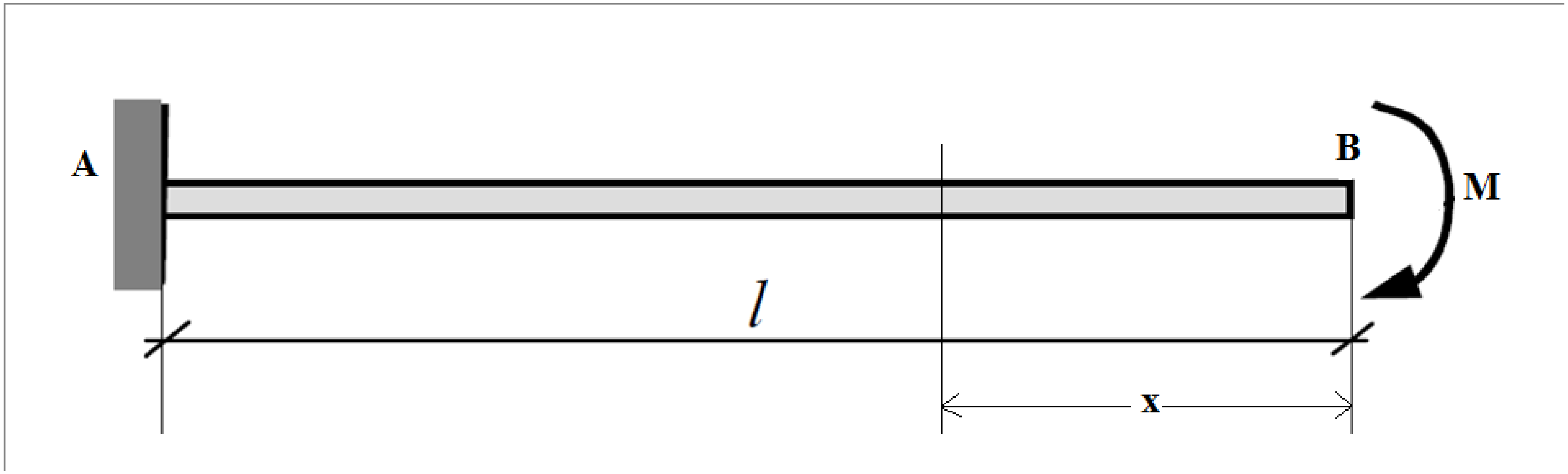
To find deflection @ free end

Put x=0 in equation II

$$EIy_B = \frac{wl^4}{30}$$

$$y_B = \frac{wl^4}{30EI}\ downward$$

Ex 8.8: A Cantilever is subjected to a moment at the free end. Derive equations for slope and deflection and find the slope and deflection at the free end.



Soln:

$$EI\ \frac{d^2y}{dx^2} = -M_x = -[-M] = M$$

$$EIy' = Mx + C_1$$

$$EIy = \frac{Mx^2}{2} + C_1\, x + C_2$$

At $x = l, y' = 0 \quad \Rightarrow \quad 0 = Ml + C_1$

$$\Rightarrow \quad C_1 = -Ml$$

At $x = l, y = 0 \quad \Rightarrow 0 = \frac{Ml^2}{2} + C_1\, l + C_2$

$$\frac{Ml^2}{2} - Ml^2 + C_2 = 0 \Rightarrow \qquad C_2 = \frac{Ml^2}{2}$$

Final equation for slope at any point is

$$EIy' = M(x - l) \longrightarrow \quad I$$

Final equation for deflection is

$$EIy = \frac{Mx^2}{2} - Mlx + \frac{Ml^2}{2} \quad \longrightarrow \quad II$$

To find θ at free end,

Put $x = 0 \; in \; equation \; I$,

$$EI\theta_B = -Ml$$

$$\theta_B = -\frac{Ml}{EI}$$

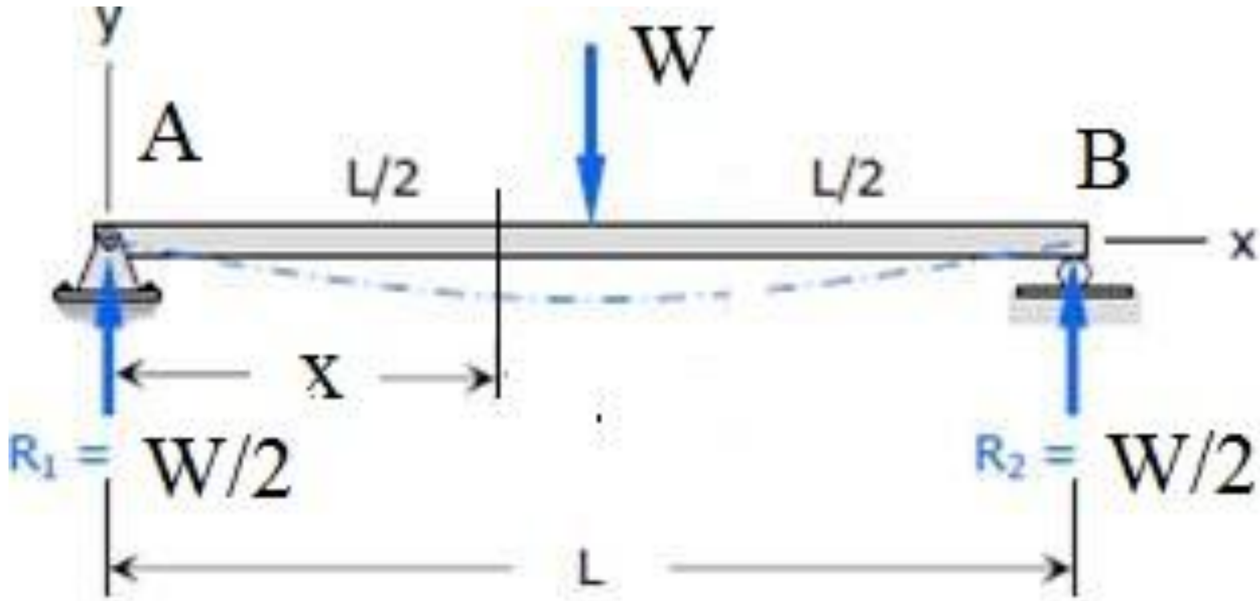
$$\theta_B = \frac{Ml}{EI} \quad clockwise$$

To find deflection at free end,

Put $x = 0 \; in \; equation \; II$,

$$y = \frac{Ml^2}{2EI} \; downward$$

Ex:69. A simply supported beam of length L carries a concentrated load W at mid span. Derive equations for slope and deflection and determine the slope at the supports and maximum deflection in the beam.



$$EIy'' = -\frac{W}{2}x$$

$$EIy' = -\frac{W}{4}x^2 + C_1$$

$$EIy = -\frac{W}{12}x^3 + C_1 x + C_2$$

At x = 0, y = 0, $\Rightarrow$ $C_2 = 0$

At $x = \frac{l}{2}$, y′ = 0 $\Rightarrow$

$$0 = -\frac{W}{4} \times \frac{L^2}{4} + C_1$$

$$C_1 = \frac{WL^2}{16}$$

- Thus, $EIy' = -\frac{W}{4}x^2 + \frac{WL^2}{16}$

and $EIy = -\frac{W}{12}x^3 + \frac{W}{16}L^2x$

Slope at A, $\theta_A = \frac{WL^2}{16EI}$ (clockwise)

Slope at B, $\theta_B = \frac{WL^2}{16EI}$ (anti-clockwise)

Maximum deflection will occur at a section where y'=0 i.e., $-\frac{W}{4}x^2 + \frac{WL^2}{16}$ = 0

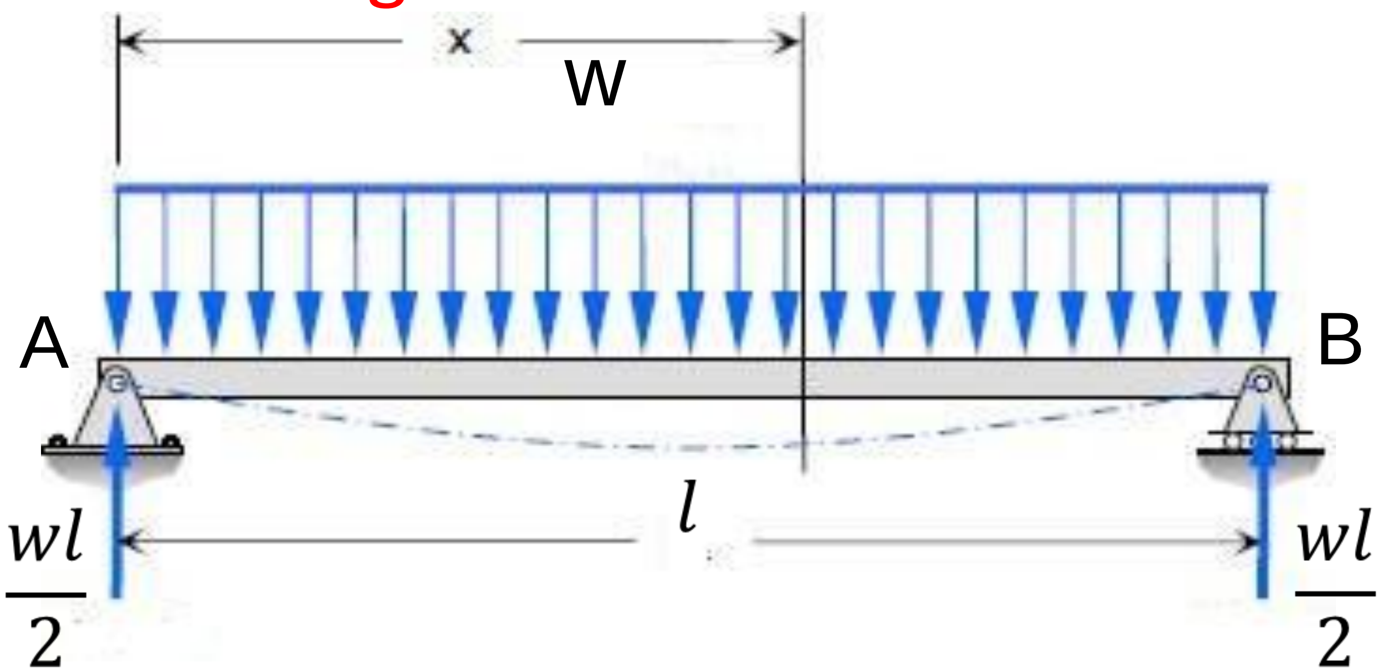
$\therefore$ x = L/2 (midspan)

$$\therefore EIy_{max} = -\frac{W}{12}\left(\frac{L}{2}\right)^3 + \frac{W}{16}L^2(\frac{1}{2}L)$$

$$EIy_{max} = -\frac{W}{96}L^3 + \frac{W}{32}L^3$$

$$y_{max} = \frac{WL^3}{48EI}$$

Ex.70: Derive equations for slope and deflection and determine the maximum deflection in a simply supported beam of length L carrying a uniformly distributed load of intensity w applied over its entire length.



$$EIy'' = -\left[w\frac{l}{2}x - wx(\frac{x}{2}) \right]$$

$$EIy'' = -w\frac{l}{2}x + \frac{w}{2}x^2$$

$$\therefore EIy' = -w\frac{l}{4}x^2 + \frac{w}{6}x^3 + C_1$$

$$and\ EIy = -w\frac{l}{12}x^3 + \frac{1}{24}wx^4 + C_1x + C_2$$

At x = 0, y = 0, therefore $C_2 = 0$

At x = L, y = 0

$$0 = -\frac{1}{12}wl^4 + \frac{1}{24}wl^4 + C_1 l$$

$$C_1 = \frac{1}{24}wl^3$$

Therefore,

$$EIy = -\frac{1}{12}wlx^3 + \frac{1}{24}wx^4 + \frac{1}{24}wl^3 x$$

Maximum deflection will occur at mid span

$$EIy_{max} = -\frac{1}{12}wl\left(\frac{l}{2}\right)^3 + \frac{1}{24}w\left(\frac{l}{2}\right)^4 + \frac{1}{24}wl^3(\frac{l}{2})$$

$$EIy_{max} = -\frac{1}{96}wl^4 + \frac{1}{384}wl^4 + \frac{1}{48}wl^4$$

$$EIy_{max} = \frac{5}{384}wl^4$$

$$\delta_{max} = \frac{5wl^4}{384EI}$$

MACAULAY’S METHOD

- Macaulay's Method:

➢ Using the double integration method to find expressions for the deflection of loaded beams, it is normally necessary to have a separate expression for the Bending Moment for each portion of the beam between adjacent concentrated loads or reactions.

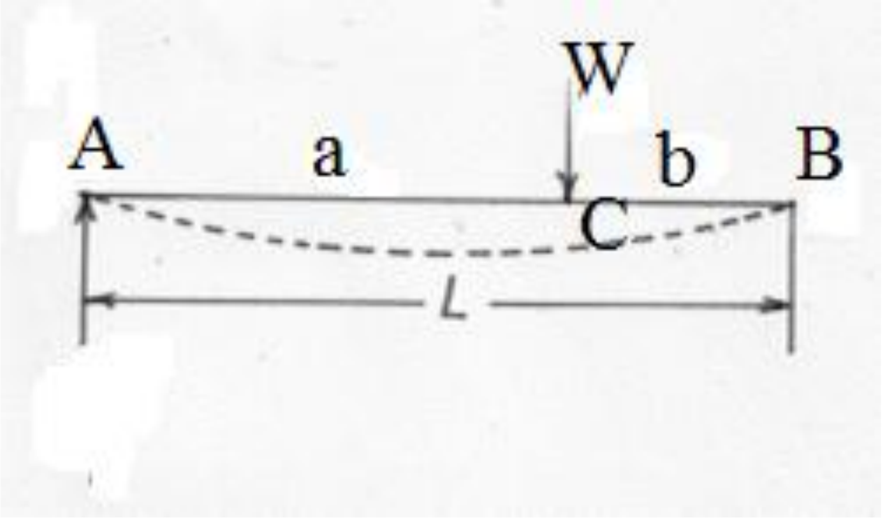
➢ Each portion will produce its own equation with its own constants of integration.

- Macaulay’s Method:

- It will be appreciated that in all but the simplest cases the work involved will be laborious, the separate equations being linked together by equating slopes and deflections given by the expressions on either side of each "junction point”.

- However a method devised by Macaulay enables one continuous expression for bending moment to be obtained and provided that certain rules are followed that the constants of integration will be the same for all sections of the beam.

- **Ex 71:** A simply supported beam of length L carries a load W at a distance ‘a’ from one end (a > b). Find the slope at the supports, slope and deflection under the load. Also find the position and magnitude of the maximum deflection.



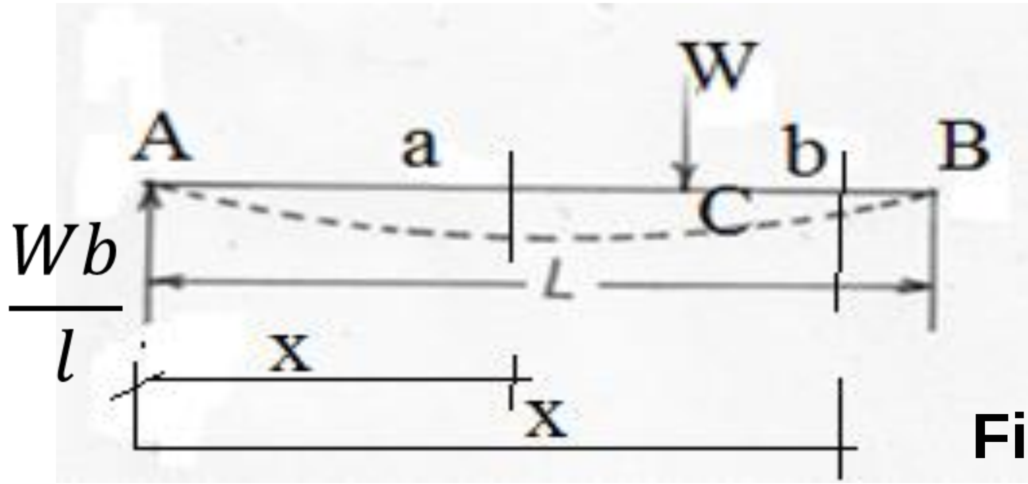


Fig. 8.14

$$EI\frac{d^2y}{dx^2} = -Mx = -\left[\left(\frac{Wb}{L}\right)x - W[x-a]\right]$$

- Integrating

$$EI\frac{dy}{dx} = -\left(\frac{Wb}{L}\right)\left(\frac{x^2}{2}\right) + C_1 + W[x-a]^2$$

(1)

- And Integrating again gives:-

$$EIy = -\left(\frac{Wb}{L}\right)\left(\frac{x^3}{6}\right) + C_1 x + C_2 + \left(\frac{W}{6}\right)[x-a]^3 \qquad (2)$$

- At x = 0; y = 0 $\Rightarrow$ $C_2 = 0$
- At x = 0; y = L $\Rightarrow$

$$C_1 L = \left(\frac{Wb}{L}\right)\left(\frac{L^3}{6}\right) - \left(\frac{W}{6}\right)b^3$$

- From which:-

$$C_1 = \left(\frac{Wb}{6L}\right)(L^2 - b^2)$$

$$\therefore EI\frac{dy}{dx} = -\left(\frac{Wb}{L}\right)\left(\frac{x^2}{2}\right) + \left(\frac{Wb}{6L}\right)(L^2 - b^2) + W[x-a]^2 \text{------------(i)}$$

and

$$EIy = -\left(\frac{Wb}{L}\right)\left(\frac{x^3}{6}\right) + \left(\frac{Wb}{6L}\right)(L^2 - b^2)x + \left(\frac{W}{6}\right)[x-a]^3 \text{-----------(ii)}$$

To find slope at A, put x = 0 in eqn (i), by considering the appropriate portion,

$$\theta_{A=}\left(\frac{Wb}{6EIL}\right)(L^2-b^2)$$

To find slope at B, put $x=l$ in eqn (i), by considering the appropriate portion,

$$EI\theta_B=-\left(\frac{Wb}{L}\right)\left(\frac{L^2}{2}\right)+\left(\frac{Wb}{6L}\right)(L^2-b^2)$$

$$+W[L-a]^2$$

$$=-\left(\frac{Wa}{6EIL}\right)(L^2-a^2)$$

To find slope at C, put $x = \text{a}$ in eqn (i),

$$EI\theta_c = -\left(\frac{Wb}{L}\right)\left(\frac{a^2}{2}\right) + \left(\frac{Wb}{6L}\right)(L^2 - b^2)$$

$$= -\frac{Wb}{6L}\lfloor 3a^2 - (a^2+2ab)\rfloor$$

$$= -\frac{\mathrm{Wab}}{3\mathrm{L}}(\mathrm{a} - \mathrm{b})$$

$$\therefore \theta_c = -\frac{\mathrm{Wab}}{3\mathrm{EIL}}(\mathrm{a} - \mathrm{b})$$

To find deflection at C, put $x = $ a in eqn (ii), we get

$$EIy_c = -\left(\frac{Wb}{L}\right)\left(\frac{a^3}{6}\right) + \left(\frac{Wb}{6L}\right)(L^2 - b^2)a$$

$$= \frac{Wab}{6L}(-a^2 + a^2 + 2ab)$$

$$= \frac{Wa^2b^2}{3L}$$

$$\therefore y_c = \frac{Wa^2b^2}{3EIL}$$

Position of maximum deflection

- We need to find the value of x where y' is zero. Using equation (i) and omitting $[x - a]$

$$\left(\frac{Wb}{L}\right)\left(\frac{x^2}{2}\right) - \left(\frac{Wb}{6L}\right)(L^2 - b^2) = 0$$

$$Hence \quad x = \sqrt{\left[\frac{L^2 - b^2}{3}\right]}$$

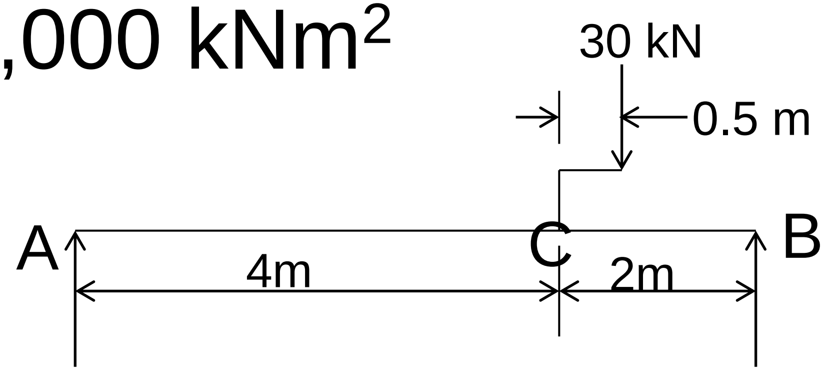
- To find the value of this deflection substitute into equation (ii)

$$EIy = -\frac{Wb}{6L} \times \frac{(L^2 - b^2)^{\frac{3}{2}}}{3\sqrt{3}} + \frac{Wb}{6L} \times \frac{(L^2 - b^2)^{\frac{3}{2}}}{\sqrt{3}}$$

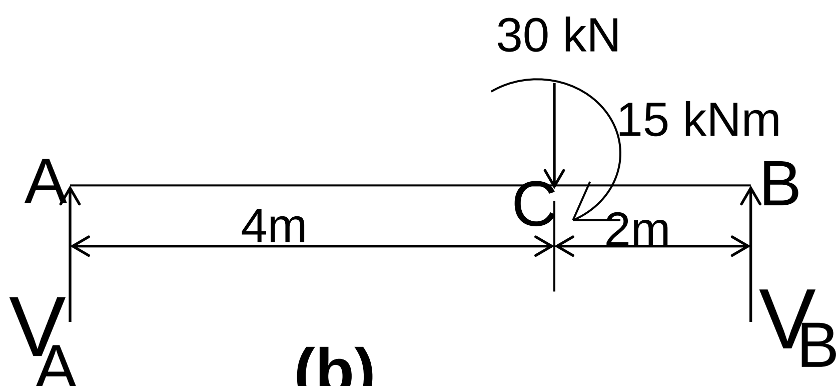
$$y = \frac{Wb(L^2 - b^2)^{\frac{3}{2}}}{9\sqrt{3}\ \times EIL}$$

Ex:72 Find the deflection at C in the beam loaded as shown in Fig. (a)

Take EI=10,000 kNm^2



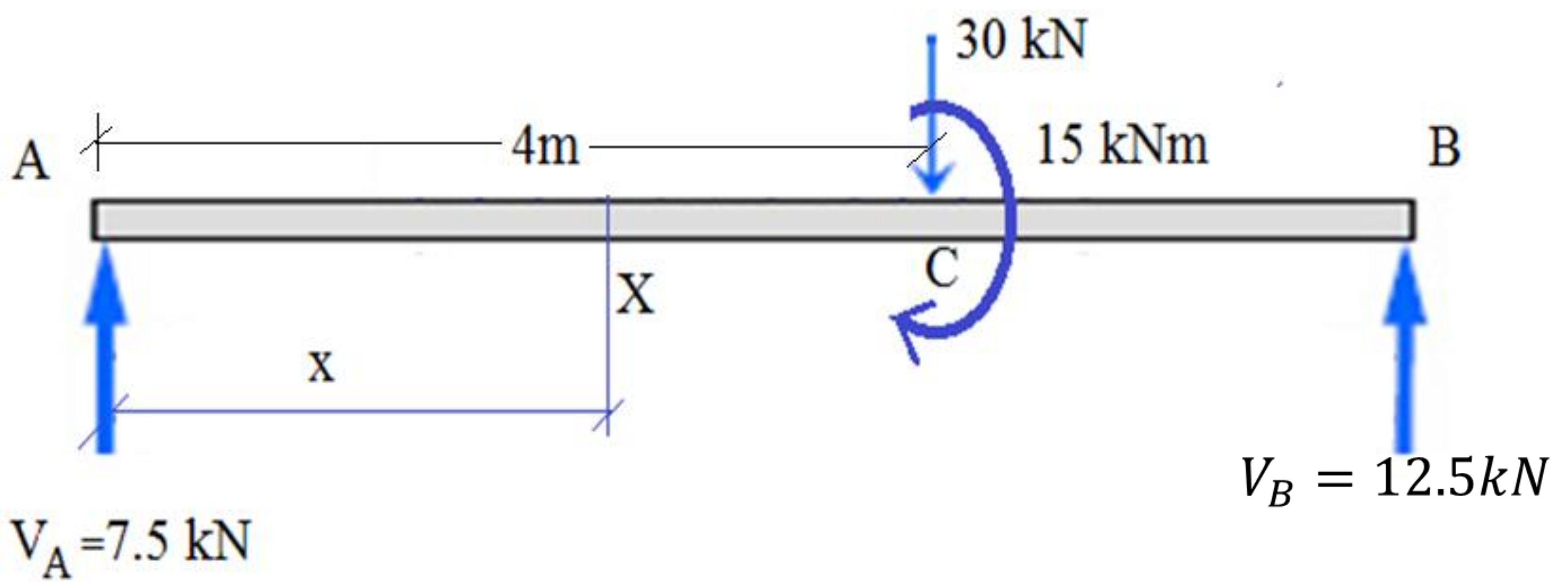
(a)



(b)

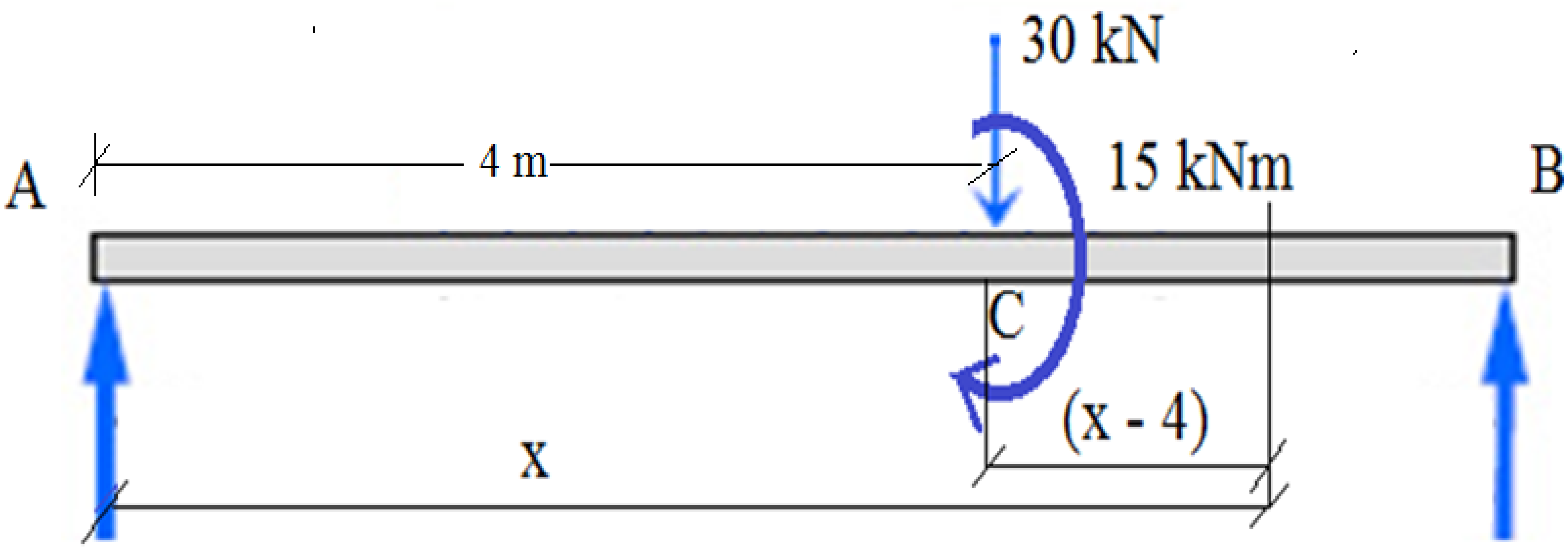
$$V_A = \frac{30 \times 2 - 15}{6} = 7.5\ kN$$

For portion AC



$$M_x = 7.5\ x\ kN$$

For portion CB



V_A=7.5 kN

$$M_x = 7.5x - 30(x-4) + 15$$

Combined equation is

$$M_x = 7.5\,x \,\big|\, - 30(x-4) + 15$$

$$EI\ \frac{d^2y}{dx^2} = -(M_x)$$

$$= -7.5x + 30(x-4) - 15(x-4)^0$$

$$EI\ \frac{dy}{dx} = -7.5\frac{x^2}{2} + C_1 + 15(x-4)^2 - 15(x-4)$$

$$EIy = -\frac{7.5}{6}x^3 + C_1x + C_2 + 5(x-4)^3$$

$$-\frac{15(x-4)^2}{2}$$

At x=0, y=0 $\Rightarrow C_2 = 0$

At x=6 m, y=0 $\Rightarrow$

$$0 = -\frac{7.5 \times 6^3}{6} + 6C_1 + 5 \times 2^3 - \frac{15 \times 2^2}{2}$$

$$C_1 = 43.333\ kNm^2$$

To find deflection at C, put x=4m in the equation for deflection by considering the relevant portion, we get

$$EIy_c = -7.5 \times \frac{4^3}{6} + 43.333 \times 4 = 93.3333\ kNm^3$$

$$i.e., \qquad 10000y_c = 93.333$$

$$y_c = 0.0093333\ m$$

$$= 9.333\ mm\ (downward\ )$$

Ex:8.13

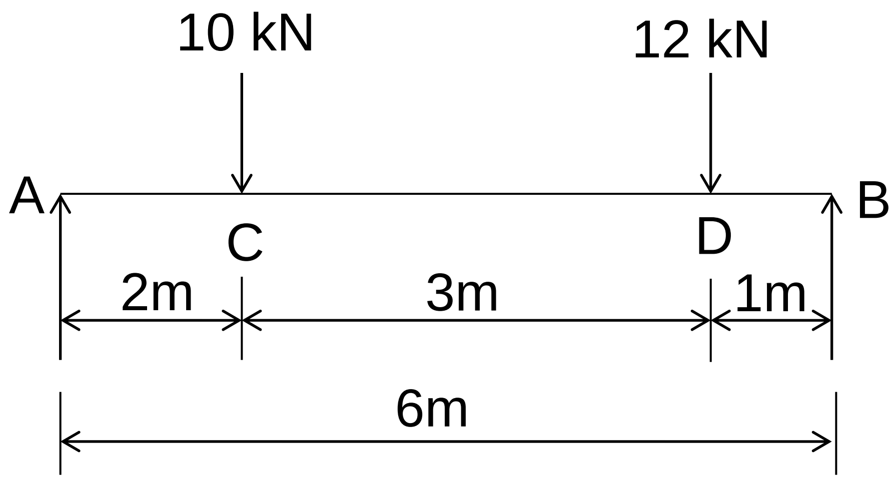


Fig. 8.18

A simply supported beam is loaded as shown in Fig.8.18. Derive equations for slope and deflection and find $\theta_A, \theta_B, \theta_C, \theta_D,\ \ y_C, y_D\ and\ y_{max}$.

Take EI=20000 kNm^2

Soln:

$$\sum M_A = 0 \Rightarrow$$

$$10 \times 2 + 12 \times 5 - V_B \times 6 = 0$$

$$V_B = 13.33\ kN$$

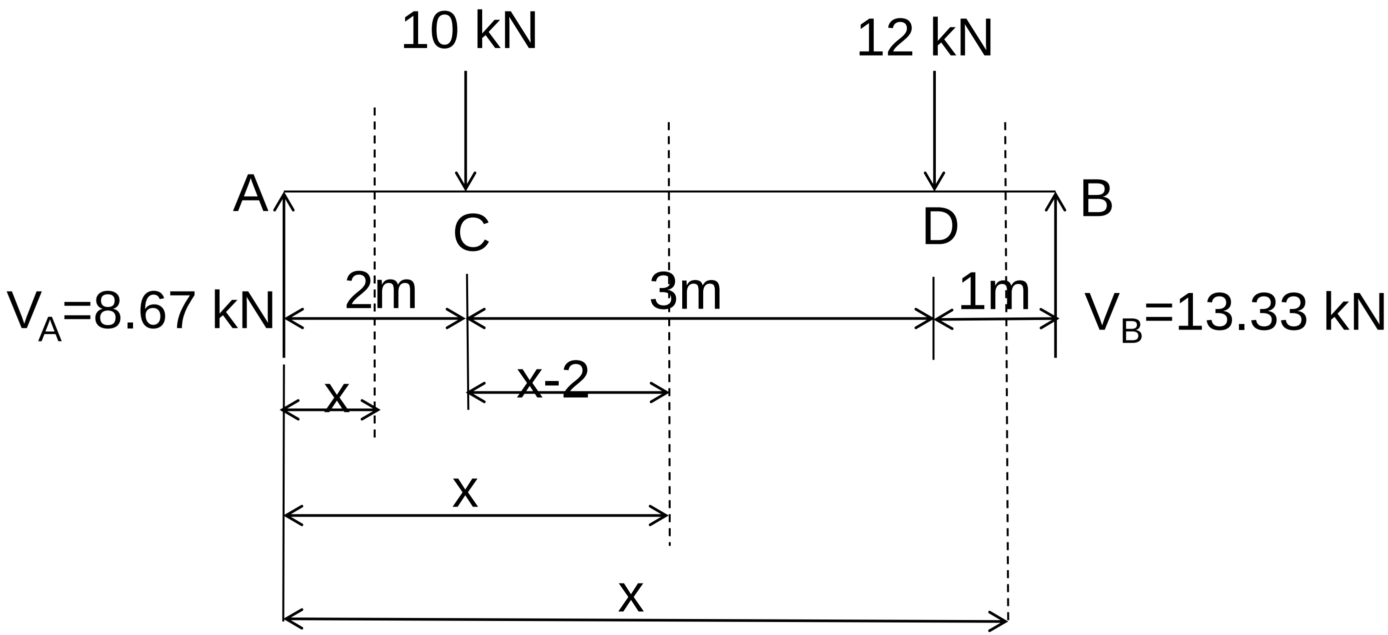
$$\sum V = 0 \Rightarrow$$

$$V_A + 13.33 - 10 - 12 = 0$$

$$V_A = 8.67\ kN$$

$$EIy'' = -M_x$$

$$= -[8.67x|-10(x-2)| - 12(x-5)]$$



$$EIy'' = -8.67x \,|\, +10(x-2) \,|\, +12(x-5)$$

$$EIy' = -\frac{8.67x^2}{2} + C_1 \left| + \frac{10(x-2)^2}{2} \right|$$

$$+\frac{12(x-5)^2}{2} \longrightarrow \quad (1)$$

$$EIy = -\frac{8.67x^3}{6} + C_1 x + C_2 \left| + \frac{10(x-2)^3}{6} \right|$$

$$+\frac{12(x-5)^3}{6} \longrightarrow \quad (2)$$

At $x = 0, y = 0 \Rightarrow C_2 = 0$

At $x = 6, y = 0 \Rightarrow$

$$0 = -\frac{8.67 \times 6^3}{6} + C_1(6) \Bigg| + \frac{10(6-2)^3}{6} \Bigg| + \frac{12(6-5)^3}{6}$$

$$C_1 = 33.91\; kNm^2$$

$$EIy' = -\frac{8.67x^2}{2} + 33.91 \Bigg| + \frac{10(x-2)^2}{2} \Bigg| + \frac{12(x-5)^2}{2}$$

$$\longrightarrow I$$

$$EIy = -\frac{8.67x^3}{6} + 33.91x \Bigg| + \frac{10(x-2)^3}{6} \Bigg| + \frac{12(x-5)^3}{6}$$

$$\longrightarrow II$$

Table 8.1

Slope	x	Value	Equation
θ_A	0	$\frac{33.91}{EI}$	I
θ_C	$2\ m$	$\frac{16.56}{EI}$	I
θ_D	$5\ m$	$\frac{-29.475}{EI}$	I
θ_B	$6\ m$	$\frac{-36.16}{EI}$	I

Table 8.2

Deflection	x	Value	Equation
y_A	0	0	II
y_C	$2\ m$	$\frac{56.24}{EI}$	II
y_D	$5\ m$	$\frac{33.875}{EI}$	II

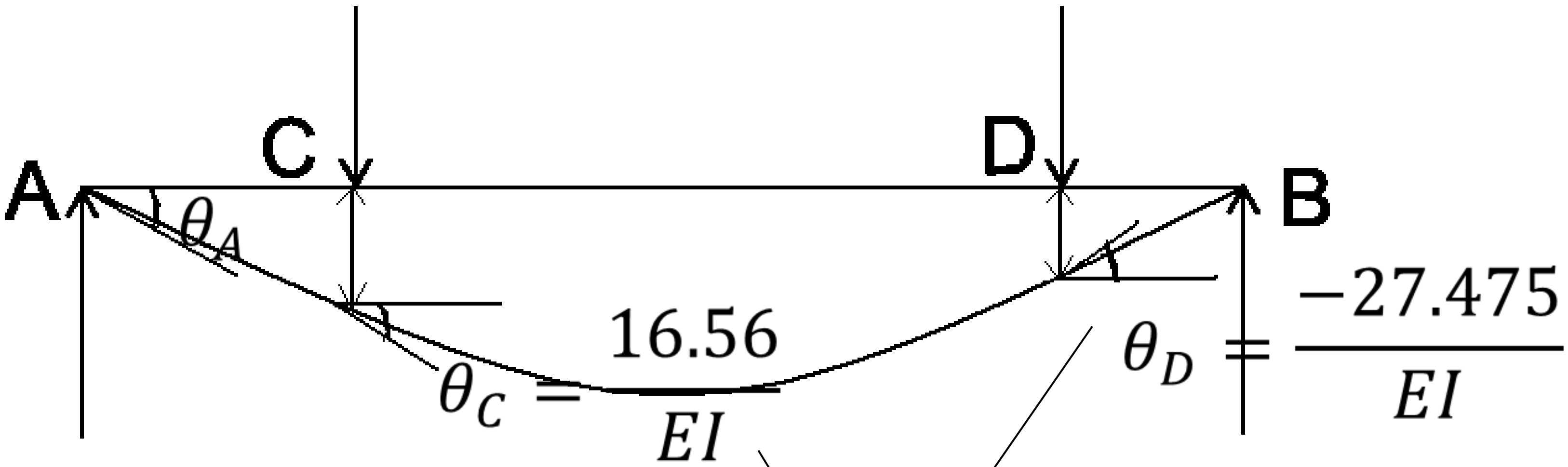


Fig. 8.20

Change of sign indicates that θ=0 lies between C and D

Maximum deflection

In simply supported beam deflection is maximum where slope is zero. From Table 8.1 and Fig.8.20, it is clear that the slope is zero between C and D

$\therefore$ equating the slope equation for CD to zero $\Rightarrow$

$$\frac{-8.67\,x^2}{2} + 33.91 + \frac{10(x-2)^2}{2} = 0$$

solving we get, $x = 2.99\ m$,

$$\therefore EI\,y_{max}$$

$$= \frac{-8.67\ (2.99)^3}{6} + 33.91(2.99)$$

$$+ \frac{10(2.99 - 2)^3}{6}$$

$$y_{max} = \frac{64.35}{EI}$$

<u>Ex:8.14</u> A beam AB is 6m long and has a second moment of area of 450×10^6 mm^4. It is supported at A and B and carries a uniformly distributed load of 10 kN/m from C to D as shown in Fig.8.22 Calculate:

i) Slope at A

ii) Deflection at mid span, and

iii) Maximum deflection

Take E=200 kN/mm^2

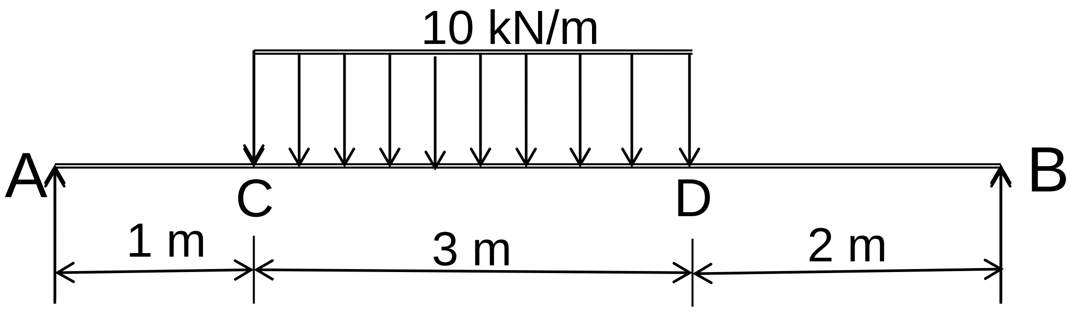
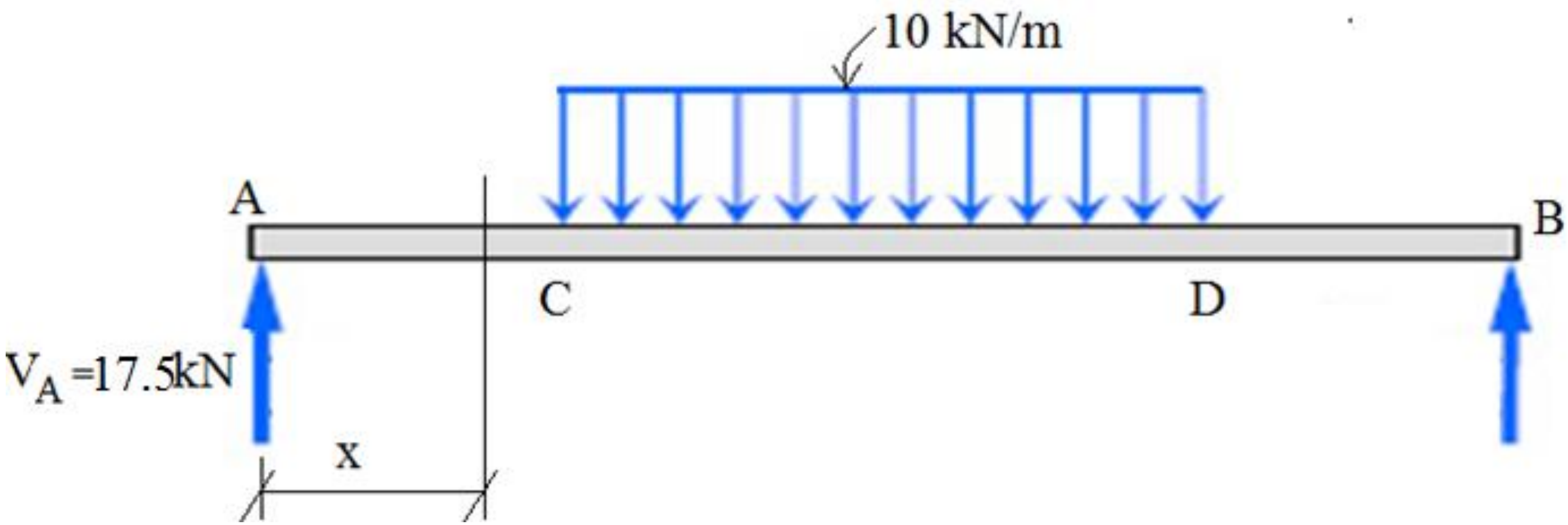


Fig. 8.22

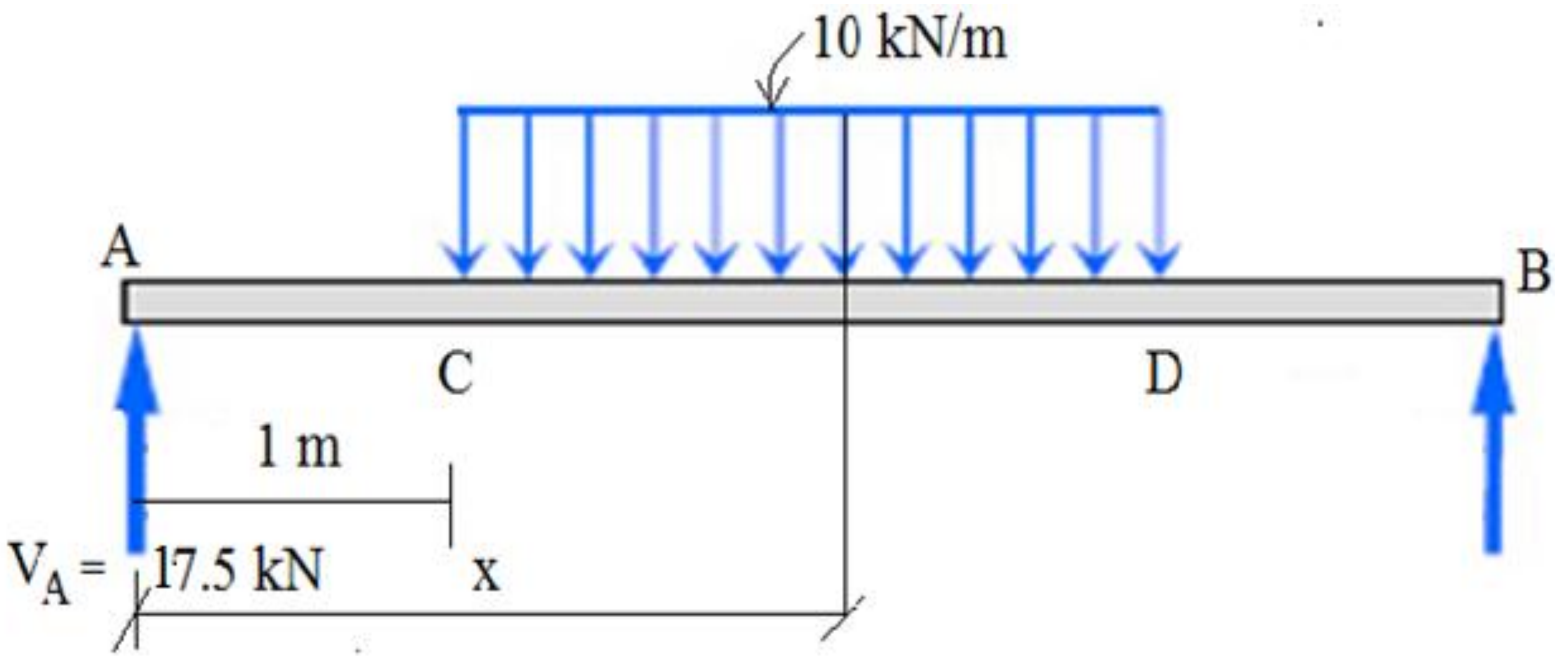
$$V_A \times 6 = 10 \times 3 \times 3.5$$
$$V_A = 17.5\ kN$$

Bending equation for AC



$$M_x = 17.5\ \mathrm{x}$$

For portion CD



$$M_x = 17.5\,x - \frac{10(x-1)^2}{2}$$

For portion DB

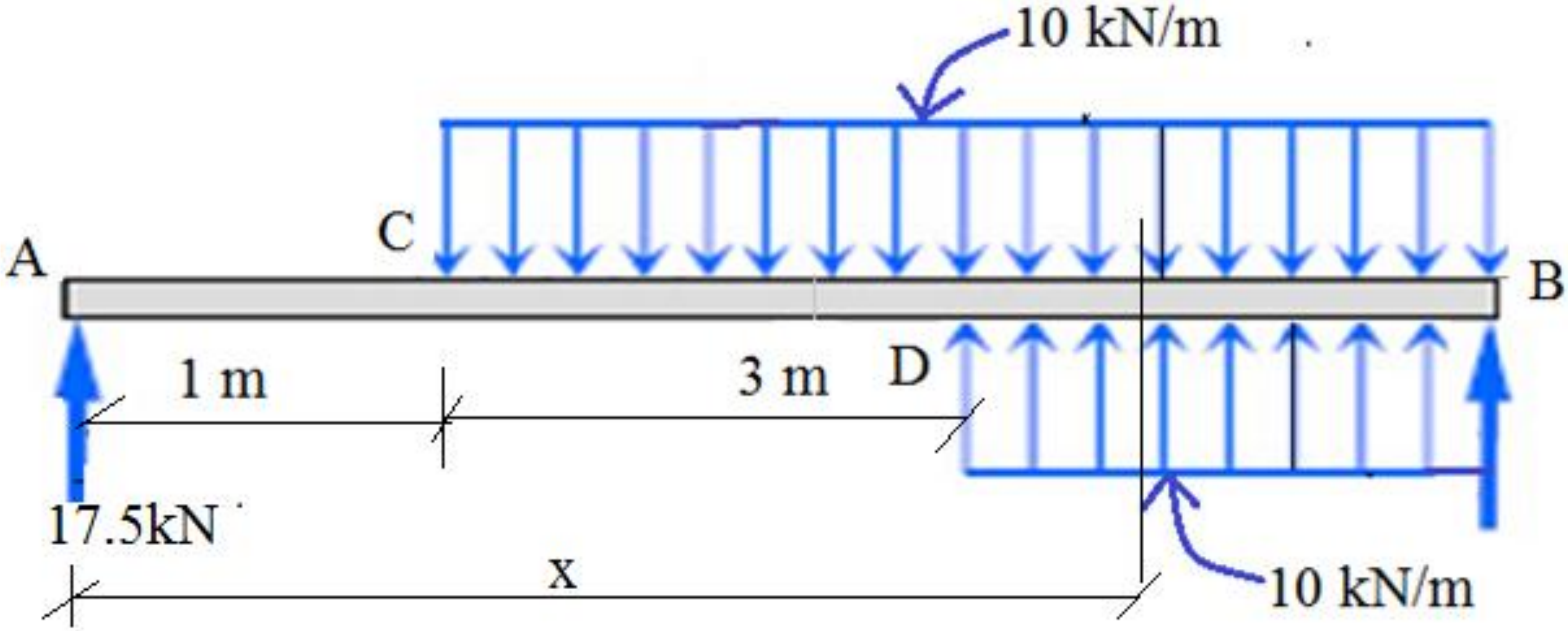


Fig. 8.25

$$M_x = 17.5\,x - \frac{10(x-1)^2}{2} + \frac{10(x-4)^2}{2}$$

To maintain the generality of the expression, as required in Macaulay’s method downward udl is extended up to right support and equal udl is applied in upward direction in portion DB as shown in Fig.8.25

$$M_x = 17.5x - \frac{10(x-1)^2}{2} + \frac{10(x-4)^2}{2}\ kNm$$

$$EI\ \frac{d^2y}{dx^2} = -17.5x + 5(x-1)^2 - 5(x-4)^2$$

$$EI\,\frac{dy}{dx} = -17.5\,\frac{x^2}{2} + C_1 + \frac{5}{3}(x-1)^3 - \frac{5}{3}(x-4)^3$$

$$EIy = -17.5\,\frac{x^3}{6} + C_1x + C_2 + \frac{5}{12}(x-1)^4 - \frac{5}{12}(x-4)^4 kNm^3$$

At x=0, y=0 $\therefore\ C_2 = 0$

At x=6 m, y=0

$$0 = -17.5 \times \frac{6^3}{6} + 6C_1 + \frac{5}{12} \times 5^4 - \frac{5}{12} \times 2^4$$

$\therefore \; C_1 = 62.71 \text{ kN}m^2$

i) Slope at A i.e., at x=0

$$EI\left(\frac{dy}{dx}\right)_A = C_1 = 62.71 \text{kN}m^2$$

$$90000\left(\frac{dy}{dx}\right)_A = 62.71$$

$$\left(\frac{dy}{dx}\right)_A = 6.97 \times 10^{-4}\ rad$$

ii) Deflection at mid span i.e., at x=3 m

$$EIy_{centre} = -17.5 \times \frac{3^3}{6} + 62.71 \times 3 + \frac{5}{12} \times 2^4$$

$$= 116.05 kNm^3$$

$$90000 y_{centre} = 116.05$$

$$y_{centre} = 1.289 \times 10^{-3}\ m$$

$$= 1.289\ mm\ downward$$

iii) Maximum Deflection:

At point of maximum deflection $\frac{dy}{dx} = 0$

Assuming it to be in portion CD

$$0 = -\frac{17.5x^2}{2} + 62.71 + \frac{5}{3}(x-1)^3$$

Solving , we get x=2.92 m

$\therefore y_{max}$ *occur at* $\quad x = 2.92$ m

EIy_{max}

$$= -17.5 \times \frac{2.92^3}{6} + 62.71 \times 2.92 + \frac{5}{12} \times 1.92^4$$

$$= 116.16 kNm^3$$

$$\therefore y_{max} = \frac{116.16}{90000} = 1.29 \times 10^{-3}\ m$$

$$= 1.29mm\ (downward)$$

Ex:8.15 Plot the elastic curve and find the maximum deflection and maximum slope for the cantilever beam loaded as shown in Fig.8.26 Take $E=200$ kN/mm^2 and $I=300 \times 10^6$ mm

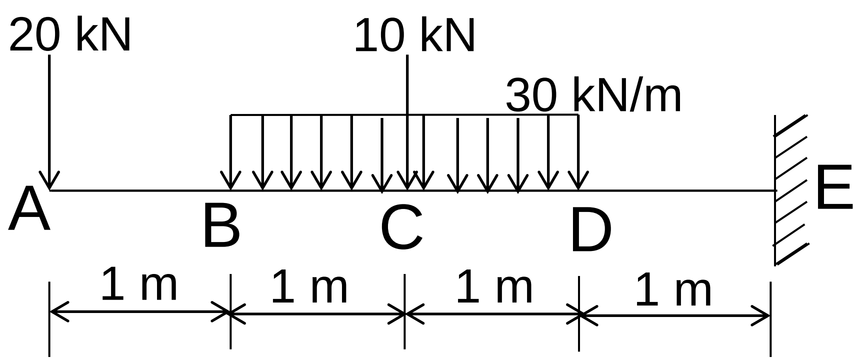


Fig. 8.26

Soln: Fig.8.27 shows the cantilever subjected to the equivalent loading but gives general expression for moment. Measuring x from free end

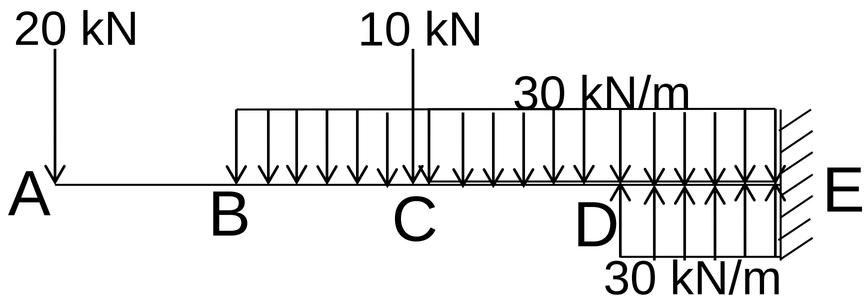


Fig. 8.27

$$EI\ \frac{d^2y}{dx^2} = -(Mx)$$

$$= -\left(-20x - \frac{30(x-1)^2}{2} - 10(x-2) + \frac{30(x-3)^2}{2}\right)$$

$$EI\ \frac{dy}{dx} = 10x^2 + C_1 + 5(x-1)^3 + 5(x-2)^2 - 5(x-3)^3$$

$$EIy = 10\ \frac{x^3}{3} + C_1x + C_2 + \frac{5}{4}(x-1)^4 + \frac{5}{3}(x-2)^3 - \frac{5}{4}(x-3)^4$$

X=4 m, $\frac{dy}{dx} = 0$

$$0 = 10 \times 4^2 + C_1 + 5 \times 3^3 + 5 \times 2^2 - 5 \times 1^3$$

$$C_1 = -310\ kNm^2$$

X=4 m, y=0

$$0 = \frac{10 \times 4^3}{3} - 310 \times 4 + C_2 + \frac{5}{4} \times 3^4 + \frac{5}{3} \times 2^3 - \frac{5}{4} \times 1^4$$

$$C_2 = 913.333\ kNm^3$$

Maximum deflection and maximum slopes occur always at free end in case of cantilever.

$$EIy_{max} = EIy_A = 913.333\; kNm^3$$

$$60000y_{max} = 913.333$$

$$y_{max} = 15.22 \times 10^{-3}\; mm$$

$$= 15.22\; mm\; (downward)$$

$$EI\left(\frac{dy}{dx}\right)_{\text{Max}} = EI\left(\frac{dy}{dx}\right)_{x=0} = C_1 = -310kNm^2$$

$$60000\left(\frac{dy}{dx}\right)_{\text{Max}} = -310$$

$$\left(\frac{dy}{dx}\right)_{\text{Max}} = -0.00517 \quad rad$$

Deflection at x=1 m,

$$EIy_B = 913.333 - 310 \times 1 + \frac{10 \times 1^3}{3}$$

$$= 606.663\ kNm^3$$

$$60000 y_B = 606.663$$

$$y_B = 10.111 \times 10^{-3}\ m$$

$$= 10.111\ mm\ (downward)$$

Deflection at x=2 m, i.e., at C

$$EIy_C = 913.333 - 310 \times 2 + \frac{10 \times 2^3}{3} + \frac{5}{4} \times 1^4$$

$$= 321.2\ kNm^3$$

$$\therefore 60000 \times 10^6 y_C = 321.24$$

$$y_C = 5.354 \times 10^{-3}\ m$$

$$= 5.354\ mm\ (downward)$$

Deflection at D where x=3 m is given by

$$EIy_D = 913.333 - 310 \times 3 + \frac{10 \times 3^3}{3} + \frac{5}{4} \times 2^4 + \frac{5}{3} \times 1^3$$

$$= 95\ kNm^3$$

$$60000 y_D = 95$$

$$y_D = 1.583 \times 10^{-3}\ m$$

$$= 1.583\ mm\ (downward)$$

$$y_E = 0$$

Hence the elastic curve (deflected shape) is as shown in Fig.

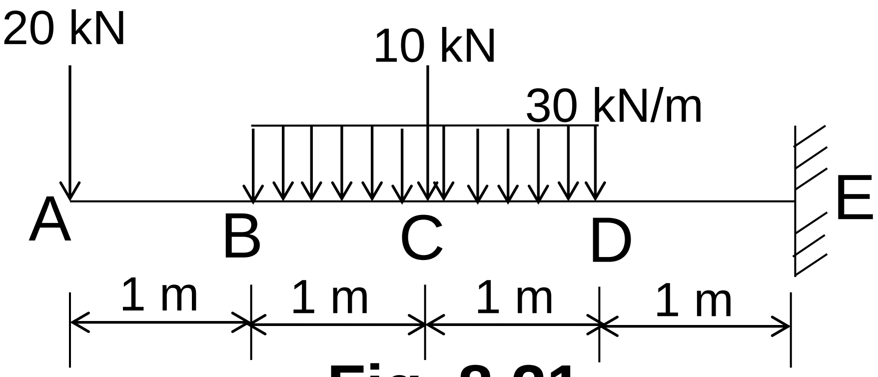


Fig. 8.21

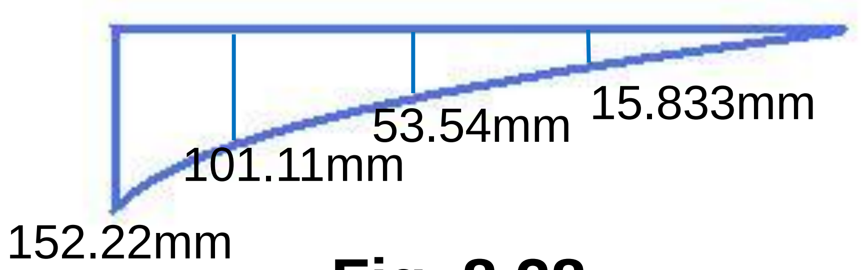


Fig. 8.28

MOMENT AREA METHOD
(MOHR'S THEOREMS)

- The moment-area method, developed by Mohr, is a powerful tool for finding the deflections of members primarily subjected to bending.

- This method is used generally to obtain displacement and rotation at a point on a beam.

MOHR'S THEOREM

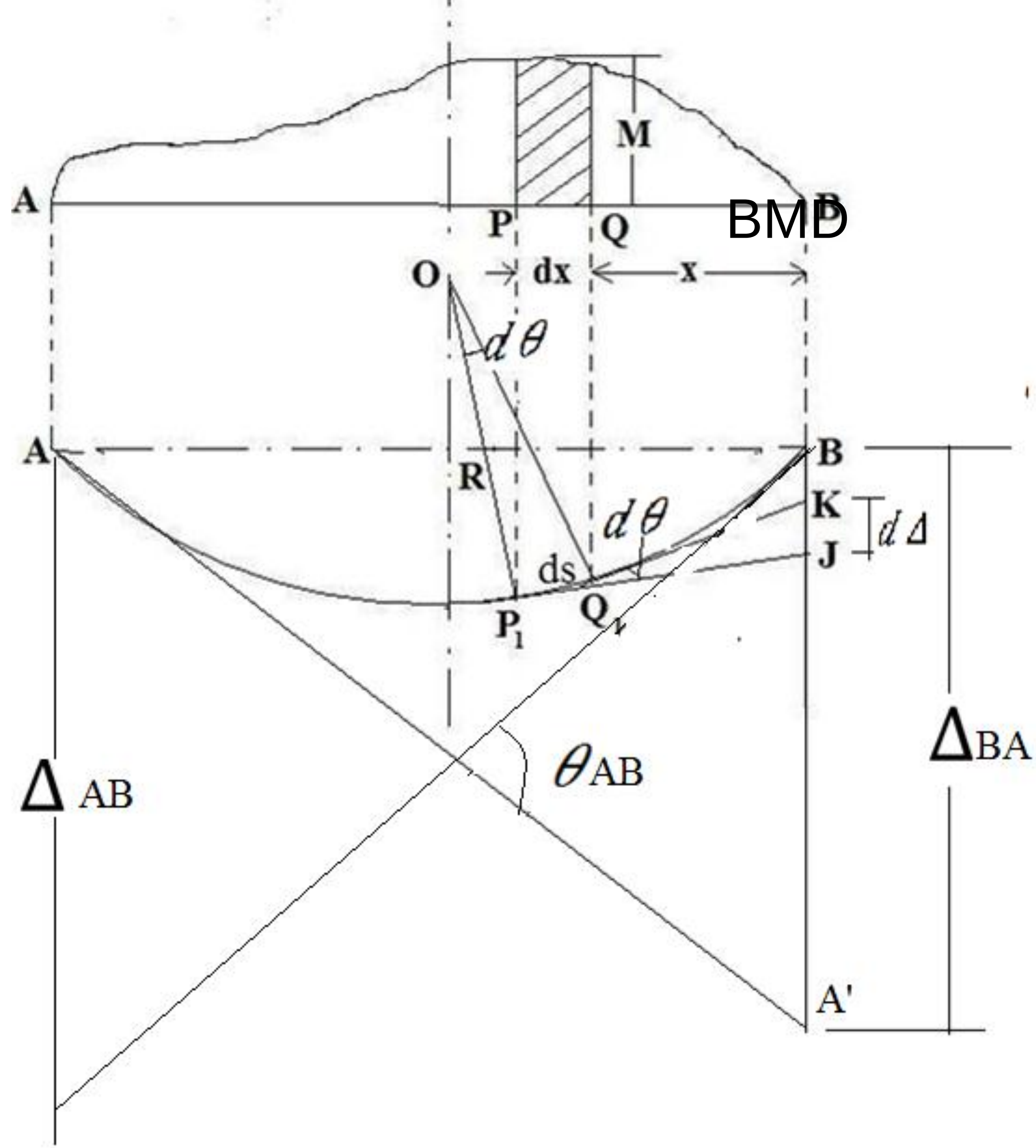


Fig.9.1

Consider a length of beam *AB* in its undeformed and deformed state, as shown in Fig 9.1

- P_1Q_1 is a very short length of the beam, measured as '*ds*' along the curve and '*dx*' along the *x*-axis.

- ‘*d*θ’ is the angle subtended at the centre of the arc ‘*ds*’.
- *M* is the average bending moment over the portion ‘*dx*’ between *P* and *Q*.
- The distance $d\Delta$ is the vertical intercept made by the two tangents drawn through P_1 and Q_1 on the vertical line passing through B.

Mohr's Theorem 1

Noting that the angles are always measured in radians, we have:

$$ds = R.\,d\theta$$

$$\therefore \qquad d\theta = \frac{ds}{R}$$

From Theory of simple Bending, we know:

$$\frac{1}{R} = \frac{M}{EI}$$

Hence:

$$d\theta = \frac{M}{EI}.\,ds$$

But for small deflections, the chord and arc length are similar, i.e. $ds \approx dx$, giving:

$$d\theta = \frac{M}{EI}.dx$$

The total change in rotation between *A* and *B* is thus:

$$\int_A^B d\theta = \int_A^B \frac{M}{EI}.dx$$

The term *M /EI* is the curvature and the diagram of this terms as it changes along a beam is the curvature diagram (or more simply the *M/EI* diagram). Thus we have:

$$d\theta_{AB} = \theta_A - \theta_B = \int_A^B \frac{M}{EI}.d\mathrm{x}$$

This is interpreted as:

$$[change\ in\ slope]_{AB} = [area\ of\ \frac{M}{EI}\ diagram]_{AB}$$

Mohr's Theorem I:

The change in slope between the tangents drawn to the elastic curve at any two points is equal to the product of 1/EI multiplied by the area of the moment diagram between those two points.

(OR)

The change in slope over any length of a member subjected to bending is equal to the area of the curvature diagram over that length.

Mohr's Theorem II

From the main diagram, we can see that the intercept made by the two tangents drawn through P_1 and Q_1 on the vertical line passing through B,

$$JK = d\Delta = x.\,d\theta$$

But, as we know from previous,

$$d\theta = \frac{M}{EI}dx$$

Thus: $$d\Delta = \frac{M}{EI}.\,x.\,dx$$

And so for the portion *AB*, we have:

$$\int_A^B d\Delta = \int_A^B \frac{M}{EI}.x.dx$$

$$\Delta_{BA} = \left[\int_A^B \frac{M}{EI}.dx\right]\bar{x}$$

$$= First\ moment\ of\ \frac{M}{EI}\ diagram$$

$$between\ A\ and\ B, about\ B$$

$$[Vertical\ Intercept]_{BA}$$

$$= \left[Area\ of\ \frac{M}{EI}\ diagram\right]_{BA}$$

$$\times \left[Distance\ from\ B\ to\ centroid\ of\ \left(\frac{M}{EI}\right)_{BA}\right] diagram$$

Mohr’s Theorem II:

The deviation of any point B relative to the tangent drawn to the elastic curve at any other point A, in a direction perpendicular to the original position of the beam, is equal to the product of 1/EI multiplied by the moment of an area about B of that part of the moment diagram between points A and B.

(OR)

For an originally straight beam, subject to bending moment, the vertical intercept between the tangent to the curve of one terminal and the tangent to the curve of another terminal is the first moment of the curvature diagram about the terminal where the intercept is measured.

Vertical intercept is *not* deflection

That is: $\Delta \neq y$

The moment of the curvature diagram must be taken about the point where the vertical intercept is required. That is:

$$\Delta_{BA} \neq \Delta_{AB}$$

SIGN CONVENTION

For bending moment:

sagging bending moment is positive

For slope:

when tangents are drawn at two points of the elastic curve and the angle measured from the tangent at left hand point to the tangent at the right hand point, if anti-clockwise, the angle is +ve and the total area of bending moment diagram between two points will be +ve and *vice versa*

SIGN CONVENTION

For deviation:

The intercept by the two tangents, drawn at two points on the deflection curve, on the vertical taken at a point, if below the deflection curve will be taken as +ve and the total moment of area of bending moment diagram between the two points about the vertical will be +ve and *vice versa*

Sign convention for slope and deviation

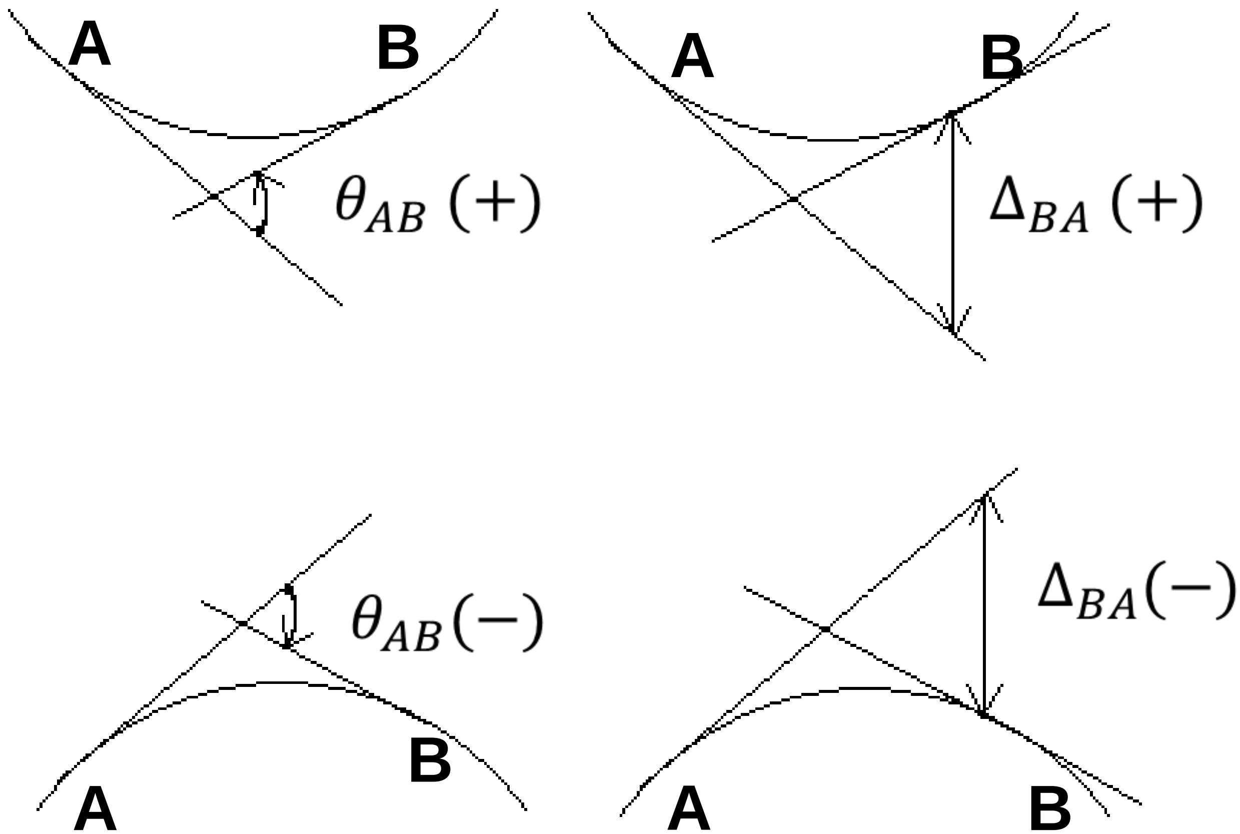
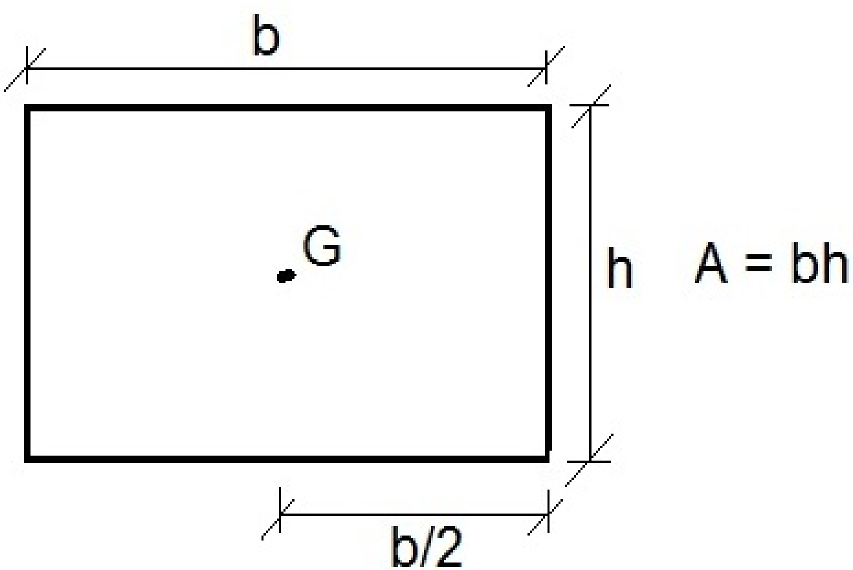
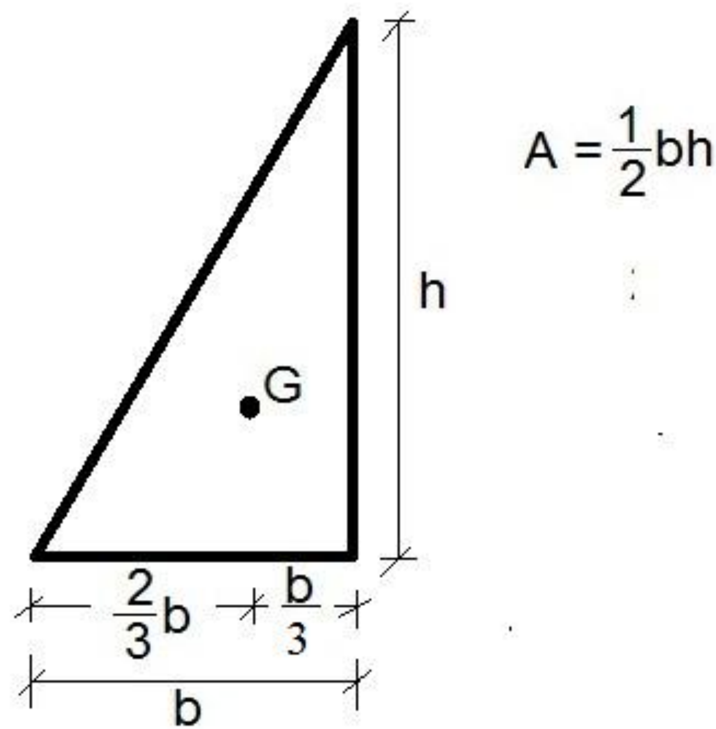
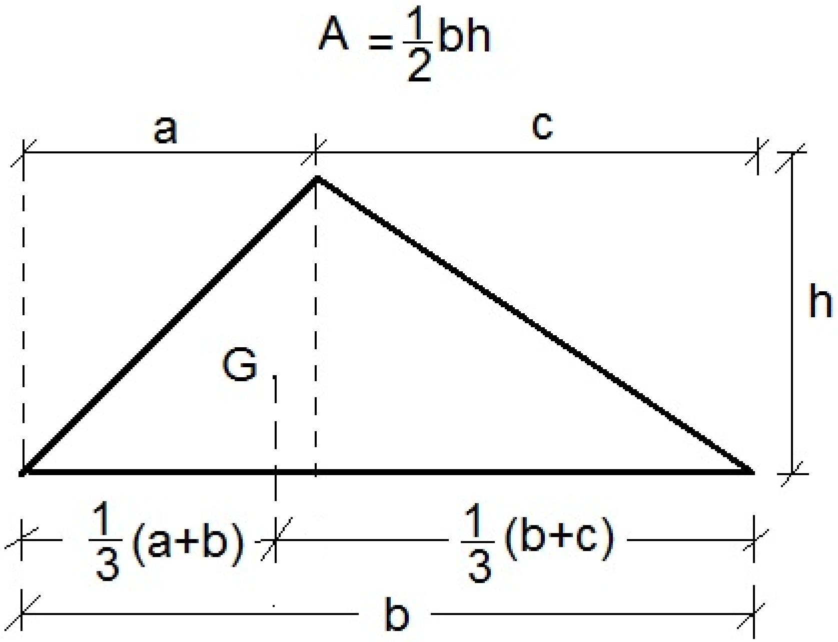


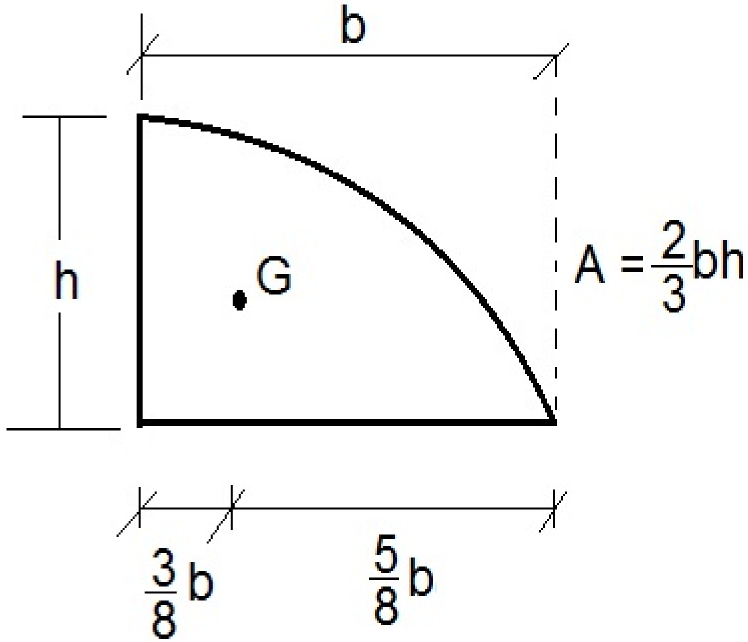
Fig.9.2

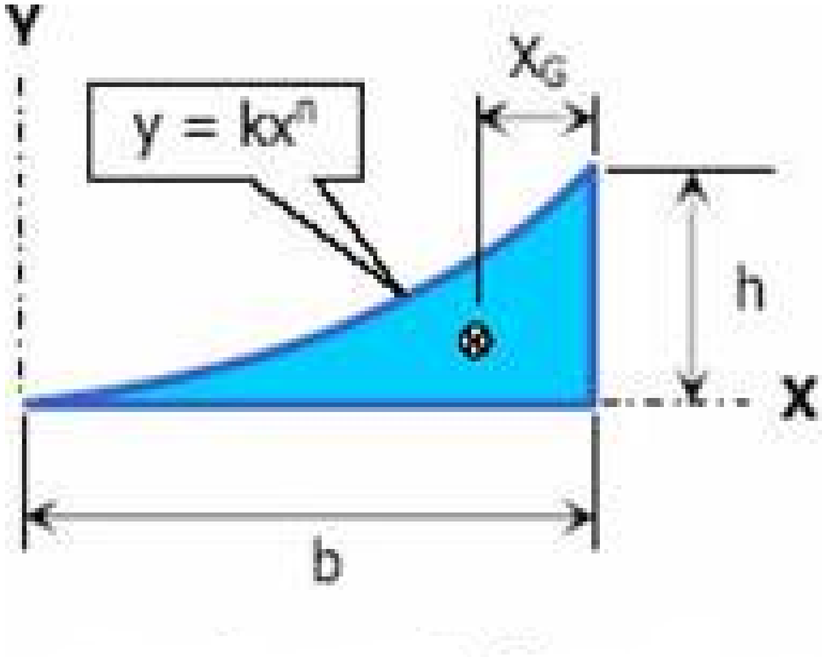
Expressions for Area and horizontal distance of C.G of some common familiar BM diagrams

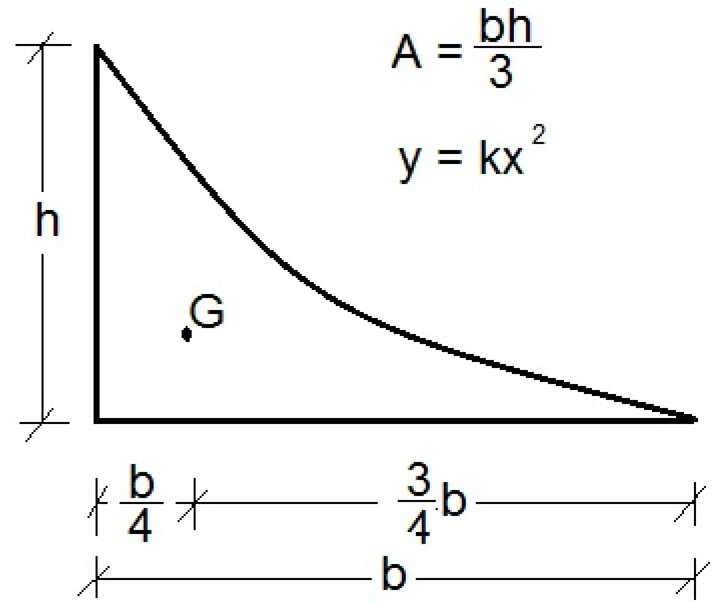





$A = \frac{1}{2}bh$
a
c
h
G
$\frac{1}{3}(a+b)$
$\frac{1}{3}(b+c)$
b

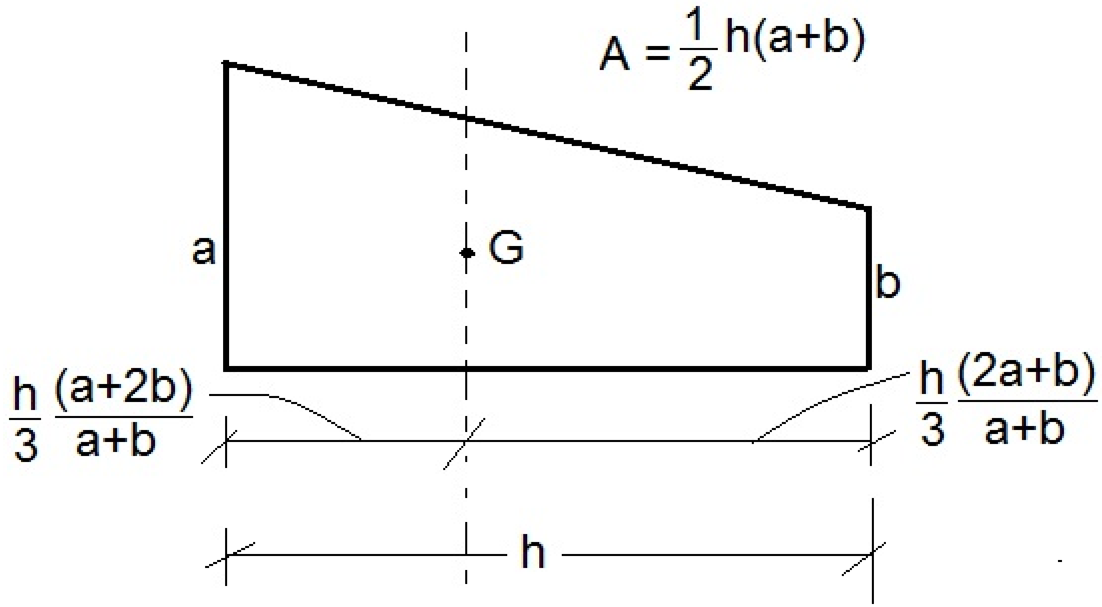

b
h
G
$A = \frac{2}{3}bh$
$\frac{3}{8}b$
$\frac{5}{8}b$

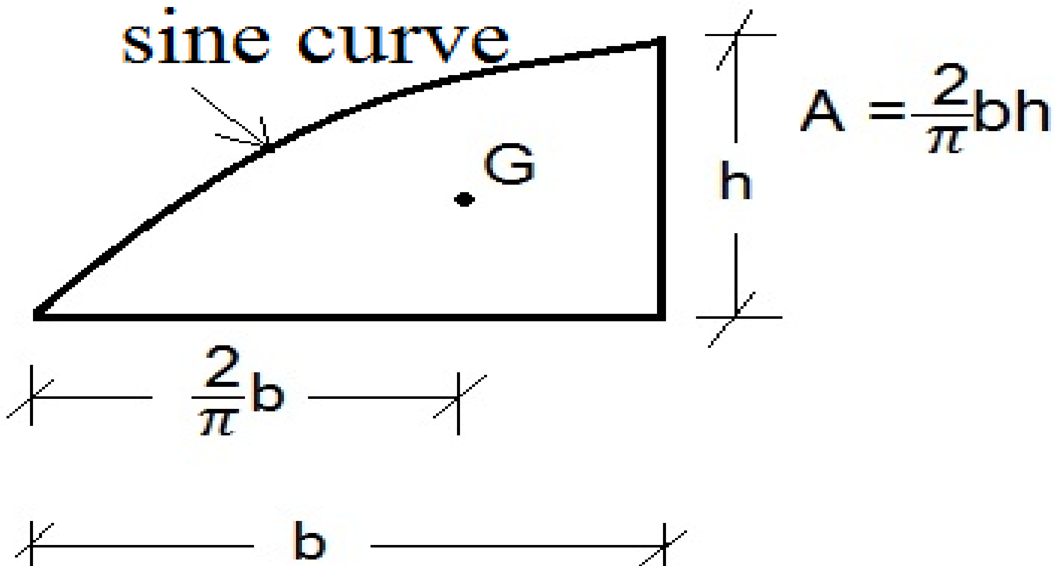




$$A = \frac{bh}{n+1}$$

$$X_G = \frac{b}{n+2}$$


A = 1/2 h(a+b)
a
G
b
h/3 (a+2b)/(a+b)
h/3 (2a+b)/(a+b)
h

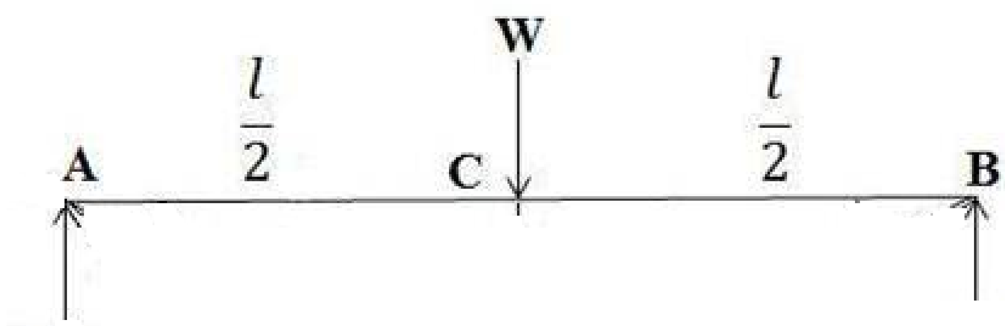

sine curve
G
h
A = 2/π bh
2/π b
b

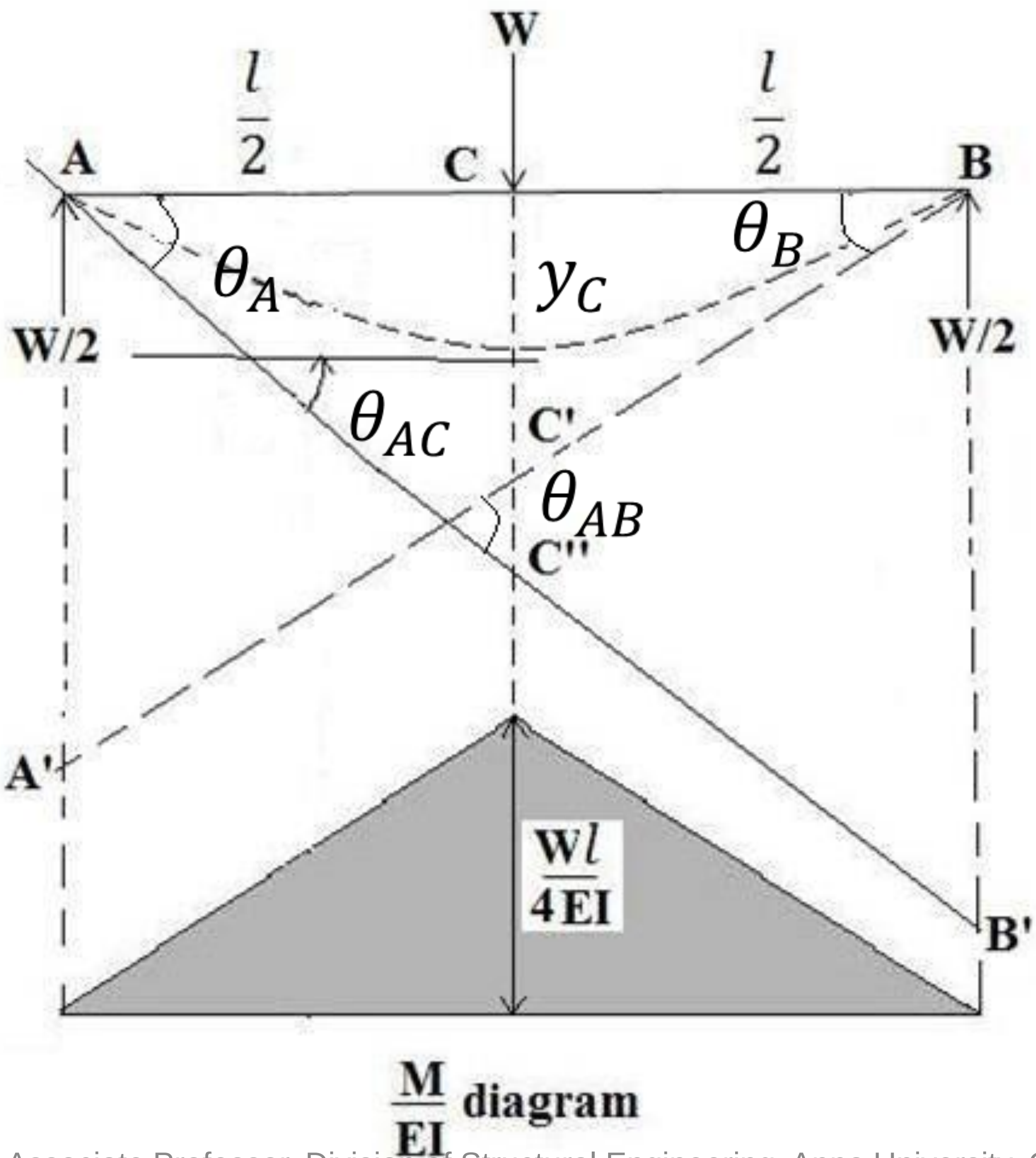
Ex:9.1 A simply supported beam is subjected to a central concentrated load W.

(i) determine the slope at the supports A, B (θ_A, θ_B)

(ii) deflection at centre (y_C)

(iii) Derive equations for slope and deflection (θ, y)





W
$\frac{l}{2}$
$\frac{l}{2}$
A
C
B
θ_B
θ_A
y_C
W/2
W/2
θ_{AC}
C'
θ_{AB}
C''
A'
$\frac{Wl}{4EI}$
B'
$\frac{M}{EI}$ diagram

Slope at A

By Mohr's theorem I, the change in slope between A and C,

θ_{AC} = area of $\frac{M}{EI}$ diagram between A and C

$$= \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{4EI} \qquad = \frac{Wl^2}{16EI}$$

We know, $\theta_{AC} = \theta_A - \theta_C = \frac{Wl^2}{16EI}$

by symmetry θ_C=0

$$\therefore \theta_A = \frac{Wl^2}{16EI}$$

Slope at A

(or)

By Mohr's theorem II,

BB' = Moment of area of M/EI diagram

between A and B, about B

$$= \frac{1}{2} \times l \times \frac{Wl}{4EI} \times \frac{l}{2} = \frac{Wl^3}{16EI}$$

From $\Delta ABB'$

$$\theta_A l = BB'$$

$$\therefore \theta_A = \frac{Wl^2}{16EI}$$

Slope at B

By Mohr's theorem I, the change in slope between C and B,

$$\theta_{CB} = area\ of\ \frac{M}{EI}\ diagram\ between\ C\ and\ B$$

$$= \frac{Wl^2}{16EI}$$

$$i.e., \theta_C - \theta_B = \frac{Wl^2}{16EI}$$

$$we\ know\ \theta_C = 0,$$

$$\therefore \theta_B = -\frac{Wl^2}{16EI}$$

$$(-sign\ indicates\ the\ slope\ is\ anticlockwise)$$

Slope at B

(or) By Mohr's theorem II,

AA′ = Moment of area of M/EI diagram between A and B about A

$$= \frac{1}{2} \times l \times \frac{Wl}{4EI} \times \frac{l}{2} = \frac{Wl^3}{16EI}$$

From $\Delta ABB'$

$$\theta_B l = AA'$$

$$\therefore \theta_B = \frac{Wl^2}{16EI}$$

Deflection at C

By Mohr's 2nd theorem,

$$C'C'' = \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{4EI} \times \frac{l}{6} \quad = \frac{Wl^3}{96EI}$$

$$y_C = CC' = CC'' - C'C''$$

$$= \frac{Wl^2}{16EI} \times \frac{l}{2} - \frac{Wl^3}{96EI}$$

$$= \frac{Wl^3}{48EI}$$

(or)

$$y_C = CC' = moment\ of\ area\ of\ \frac{M}{EI}\ diagram$$

$$between\ A\ and\ C\ about\ C$$

$$= \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{4EI} \times \frac{2}{3} \times \frac{l}{2}$$

$$= \frac{Wl^3}{48EI}$$

Slope at any point X (θ_X)

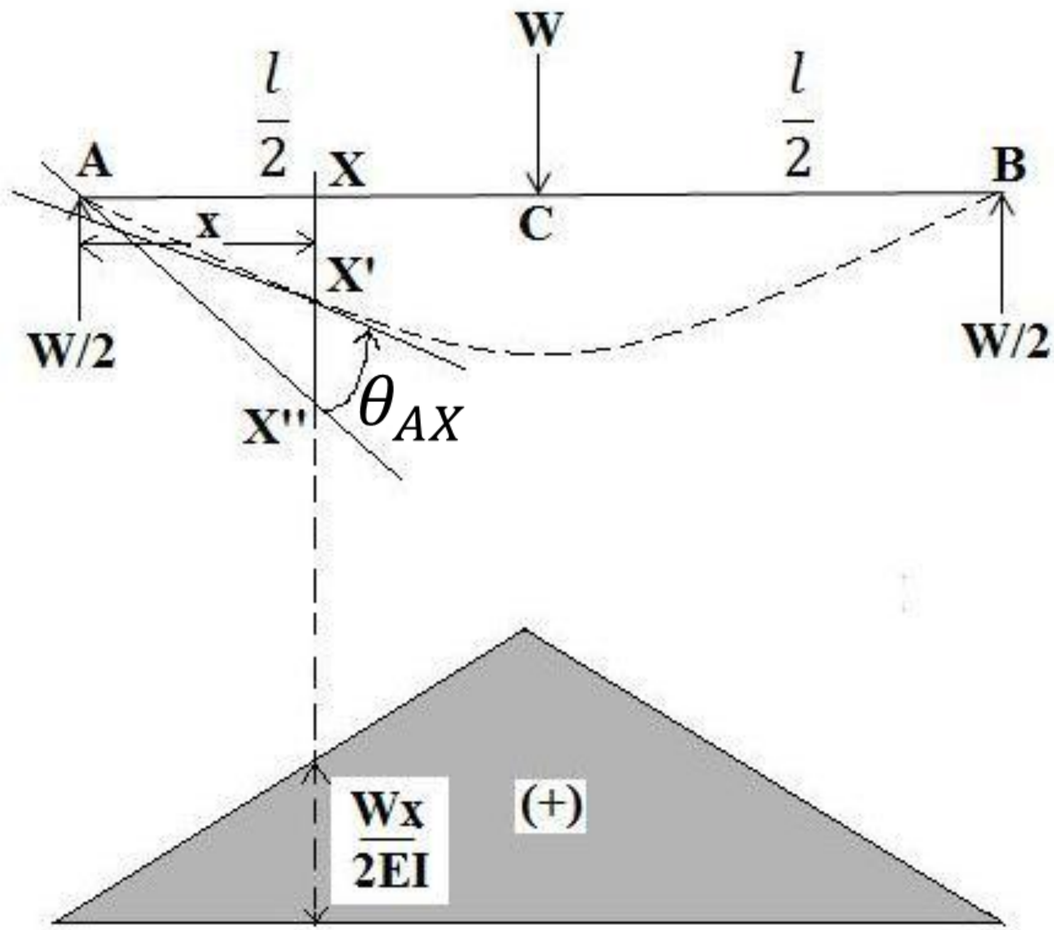
$$\theta_{AX} = \text{area of} \frac{\text{M}}{\text{EI}} \text{diagram between A and X}$$

$$= \frac{1}{2} \times x \times \frac{Wx}{2EI}$$

$$= \frac{Wx^2}{4EI}$$

$$\theta_A - \theta_X = \frac{Wx^2}{4EI}$$

$$\theta_X = \theta_A - \frac{Wx^2}{4EI} = \frac{Wl^2}{16EI} - \frac{Wx^2}{4EI}$$



$\frac{M}{EI}$ diagram

Deflection at any point X (y_X)

$$y_X = XX' = XX'' - X'X''$$

$$= \frac{Wl^2}{16EI}x - \frac{1}{2} \times x \times \frac{Wx}{2EI} \times \frac{x}{3}$$

$$= \frac{Wl^2}{16EI}x - \frac{Wx^3}{12EI}$$

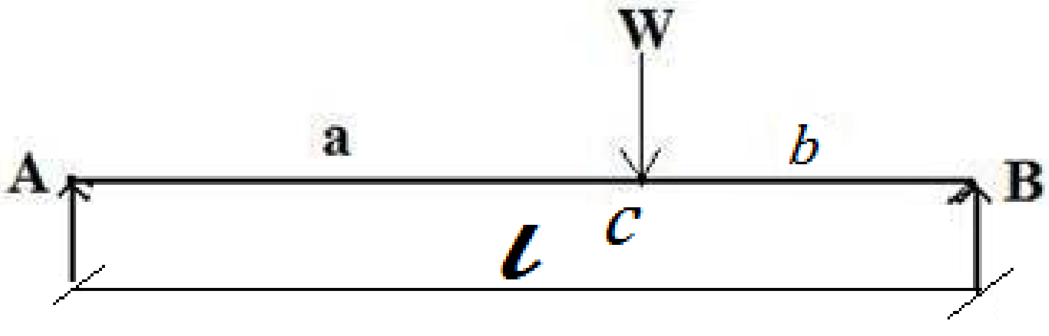
$$\therefore y_X = \frac{Wl^2}{16EI}x - \frac{Wx^3}{12EI}$$

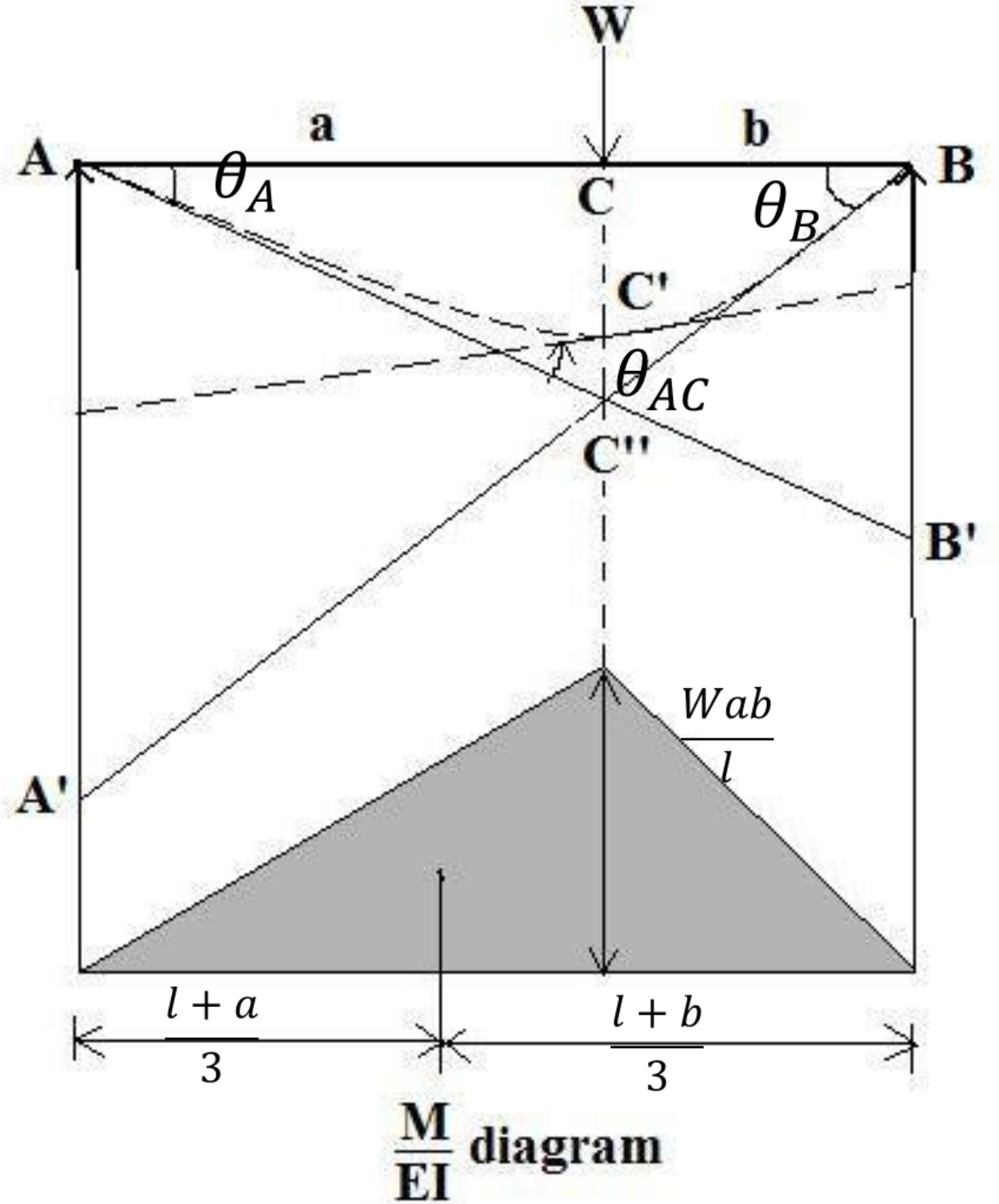
Ex:9.2 A simply supported beam is subjected to a non central concentrated load W at a distance 'a' from left end A. Determine

(i) the slope at A, B and C (θ_A, θ_B & θ_C)

(ii) deflection at C (y_C) and

(iii) Derive equations for slope and deflection (θ, y).




W
a
b
A
B
C
θ_A
θ_B
C'
θ_{AC}
C''
B'
A'
$\frac{Wab}{l}$
$\frac{l+a}{3}$
$\frac{l+b}{3}$
$\frac{M}{EI}$ diagram

Slope at A

By Mohr's theorem II,

BB' = Moment of area of M/EI diagram between A and B about B

$$= \frac{1}{2} \times l \times \frac{Wab}{lEI} \times \left(\frac{l+b}{3}\right)$$

$$= \frac{Wab}{6EI}(l + b)$$

$$\therefore \theta_A = \frac{BB'}{l} = \frac{Wab}{6EIl}(l + b)$$

Slope at B

By Mohr's theorem II,,

$$AA' = Moment\ of\ area\ of\ \frac{M}{EI}\ diagram$$

$$between\ A\ and\ B\ about\ A$$

$$= \frac{1}{2} \times l \times \frac{Wab}{lEI} \times \left(\frac{l+a}{3}\right) = \frac{1}{2}\frac{Wab}{EI}\left(\frac{l+a}{3}\right)$$

$$= \frac{Wab}{6EI}(l + a)$$

$$\therefore\ \theta_B = \frac{AA'}{l} = \frac{Wab}{6EI}(l + a)$$

Slope at C (θ_C),

$$\theta_{AC} = change\ in\ slope\ between\ A\ and\ C$$

$$= Area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ C$$

$$= \frac{1}{2} \times a \times \frac{Wab}{lEI}$$

$$\theta_A - \theta_C = \frac{Wa^2b}{2EIl}$$

$$\theta_C = \theta_A - \frac{Wa^2b}{2EIl} \quad = \frac{Wab}{6EIl}(l+b) - \frac{Wa^2b}{2EIl}$$

$$= \frac{Wab}{6EIl}(l+b-3a)$$

$$= \frac{Wab}{6EIl}2(b-a)$$

$$\therefore \theta_C = \frac{Wab}{3EIl}(b-a)$$

Deflection at C,(Yc)

$$y_C = CC' = CC'' - C'C''$$

$$= \theta_A \, a - C'C''$$

Where, $C'C'' = Moment\ of\ area\ of\ \frac{M}{EI}\ diagram$

$$between\ A\ and\ C\ about\ C$$

$$= \frac{Wa^2b}{2EIl} \times \frac{a}{3}$$

$$= \frac{Wa^3b}{6EIl}$$

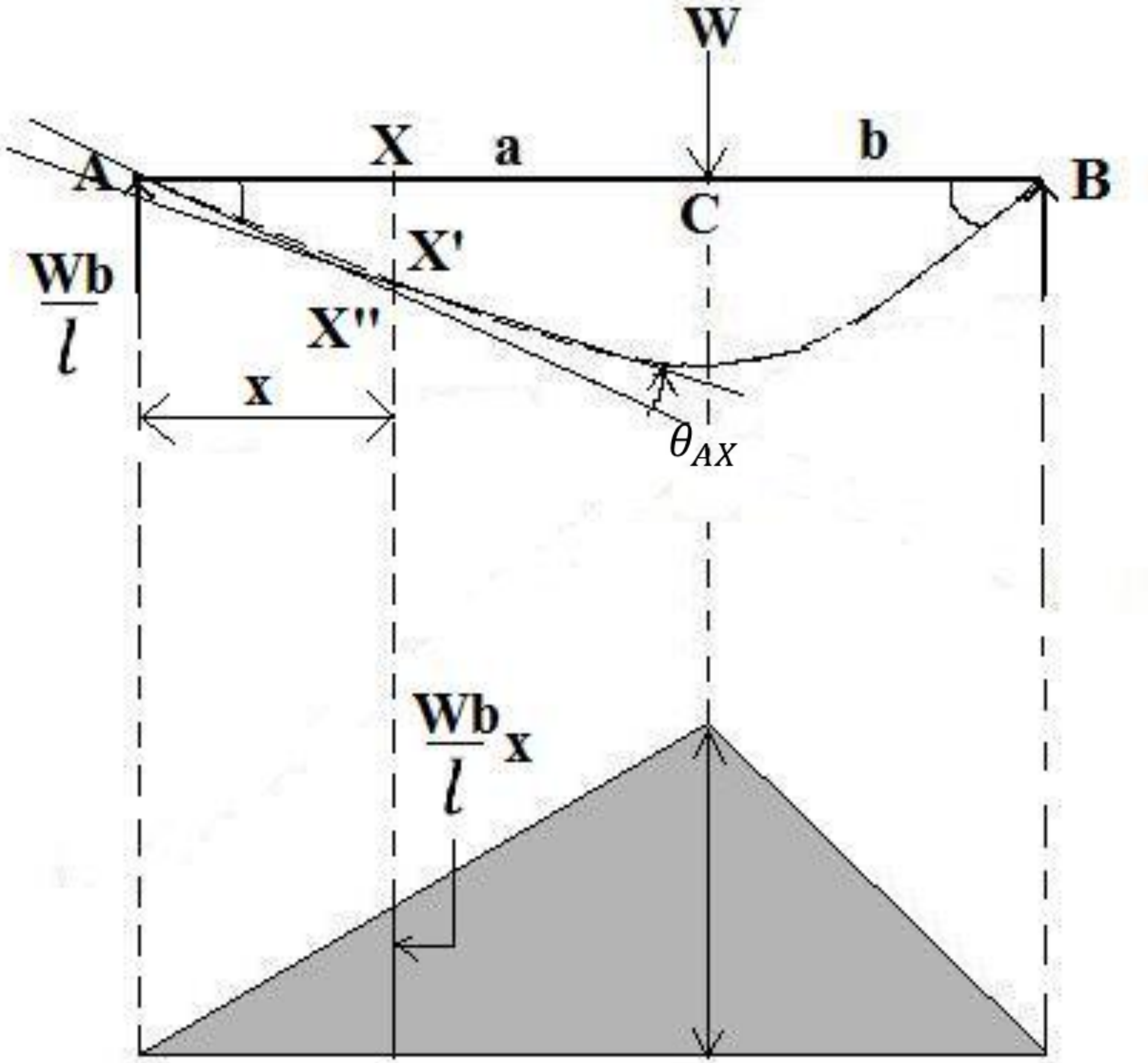
$$\therefore y_C = \frac{Wab}{6EIl}(l+b).a - \frac{Wa^3b}{6EIl}$$

$$= \frac{Wa^2b}{6EIl}(l+b-a)$$

$$= \frac{Wa^2b}{6EIl}(2b)$$

$$= \frac{Wa^2b^2}{3EIl}$$

$$\therefore y_C = \frac{Wa^2b^2}{3EIl}$$



$\frac{M}{EI}$ diagram

To find y_{max} :

$$y_X = XX'' - X'X''$$

$$= \theta_A x - moment\ of\ \frac{M}{EI}\ diagram$$

$$between\ A\ and\ X\ \ about\ X$$

$$= \frac{Wab}{6EIl}(l+b)x - \left(\frac{1}{2} \times x \times \frac{Wb}{lEI}x\right)\frac{x}{3}$$

$$= \frac{Wab}{6EIl}(l+b)x - \frac{Wbx^3}{6EIl}$$

$$= \frac{Wbx}{6EIl}(a(l+b) - x^2)$$

$$= \frac{Wbx}{6EIl}((l-b)(l+b) - x^2)$$

$$(a = l - b)$$

$$= \frac{Wbx}{6EIl}(l^2 - b^2 - x^2)$$

For y_X to be maximum, $\frac{dy_x}{dx} = 0$

$$\therefore \frac{dy_x}{dx} = \frac{Wb}{6EIl}\frac{d}{dx}(x(l^2 - b^2) - x^3)$$

$$= \frac{Wb}{6EIl}(l^2 - b^2 - 3x^2) = 0$$

$$x^2 = \frac{l^2 - b^2}{3}$$

$$\therefore x = \sqrt{\frac{l^2 - b^2}{3}}$$

$$y_{max} = \frac{Wb\left(\sqrt{\frac{l^2-b^2}{3}}\right)}{6EIl}\left(l^2 - b^2 - \frac{l^2-b^2}{3}\right)$$

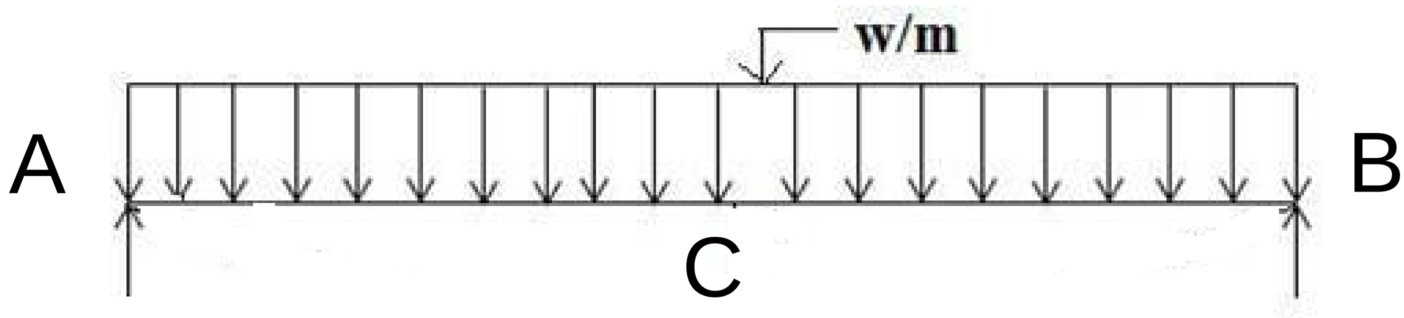
$$= \frac{Wb}{6EIl} \times \frac{2}{3} \times (l^2 - b^2)\left(\sqrt{\frac{l^2 - b^2}{3}}\right)$$

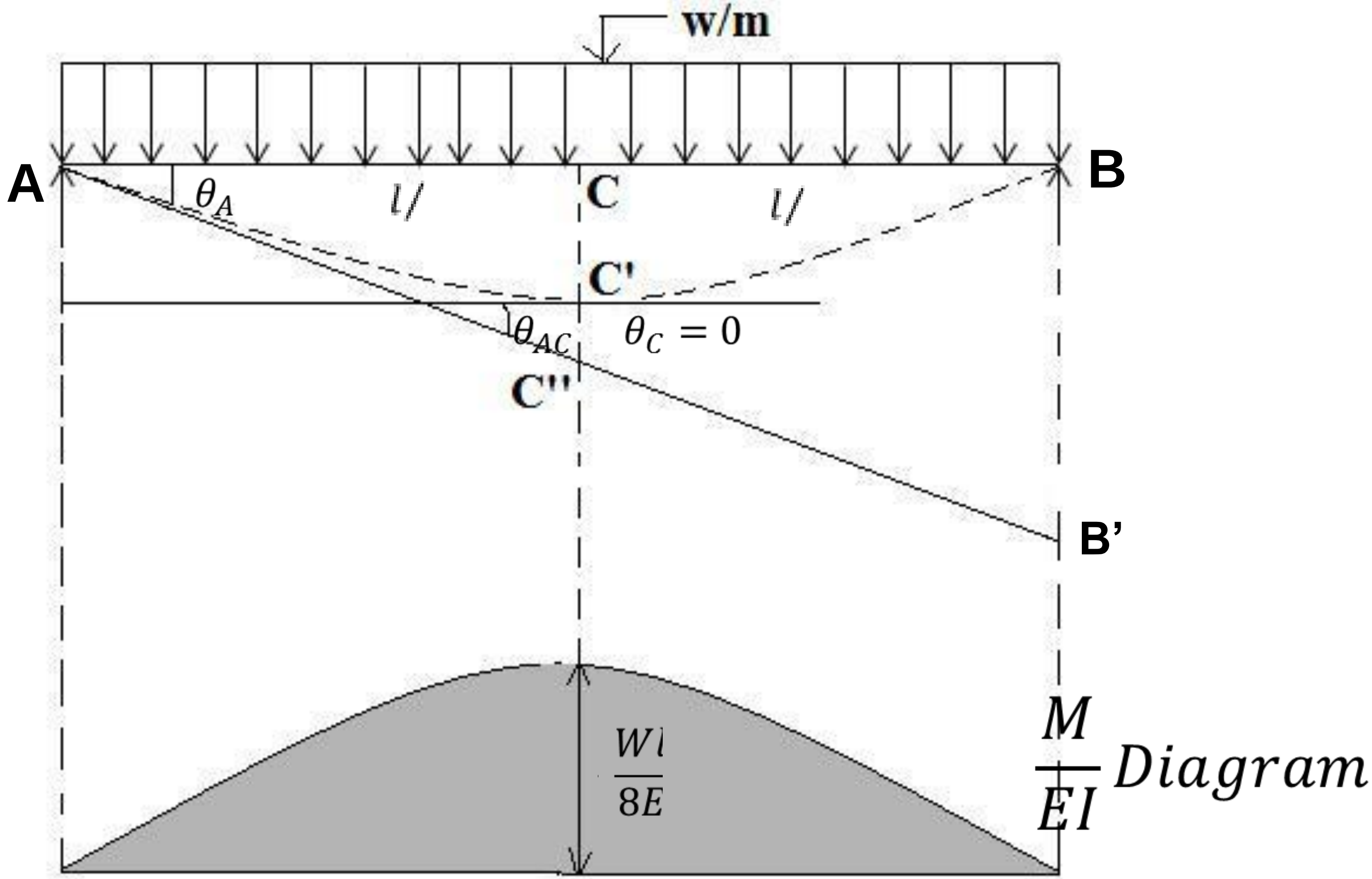
$$= \frac{Wb(l^2-b^2)^{3/2}}{9\sqrt{3}EIl}$$

Ex:9.3. A simply supported beam is subjected to a u.d.l throughout the span. Determine

(i) the slope at A & B (θ_A &θ_B)

(ii) deflection at Mid $span\ C\ (y_C)$




w/m
A
B
θ_A
$l/$
C
$l/$
C'
θ_{AC}
$\theta_C = 0$
C''
B'
$\frac{Wl}{8E}$
$\frac{M}{EI}$ $Diagram$

Slope at A and B

$$\theta_A = \frac{BB'}{l} = \frac{\left(\frac{2}{3} \times l \times \frac{Wl^2}{8EI}\right) \times \frac{l}{2}}{l}$$

$$= \frac{Wl^3}{24EI}$$

$$\theta_B = \frac{Wl^3}{24EI} = \theta_A$$ **(due to symmetry)**

(or)

$$\theta_{AC} = area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ C$$

$$= \frac{2}{3} \times \frac{l}{2} \times \frac{Wl^2}{8EI}$$

$$= \frac{Wl^3}{24EI}$$

$$\theta_{AC} = \theta_A - \theta_C = \theta_A \qquad (\theta_C = 0)$$

$$\therefore\ \theta_A = \frac{Wl^3}{24EI}$$

Deflection at C

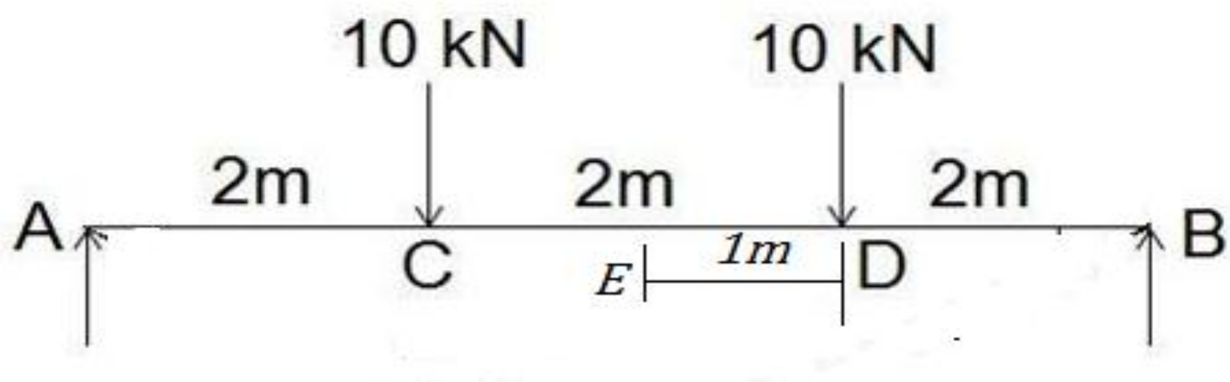
$$y_C = CC' = CC'' - C'\,C''$$

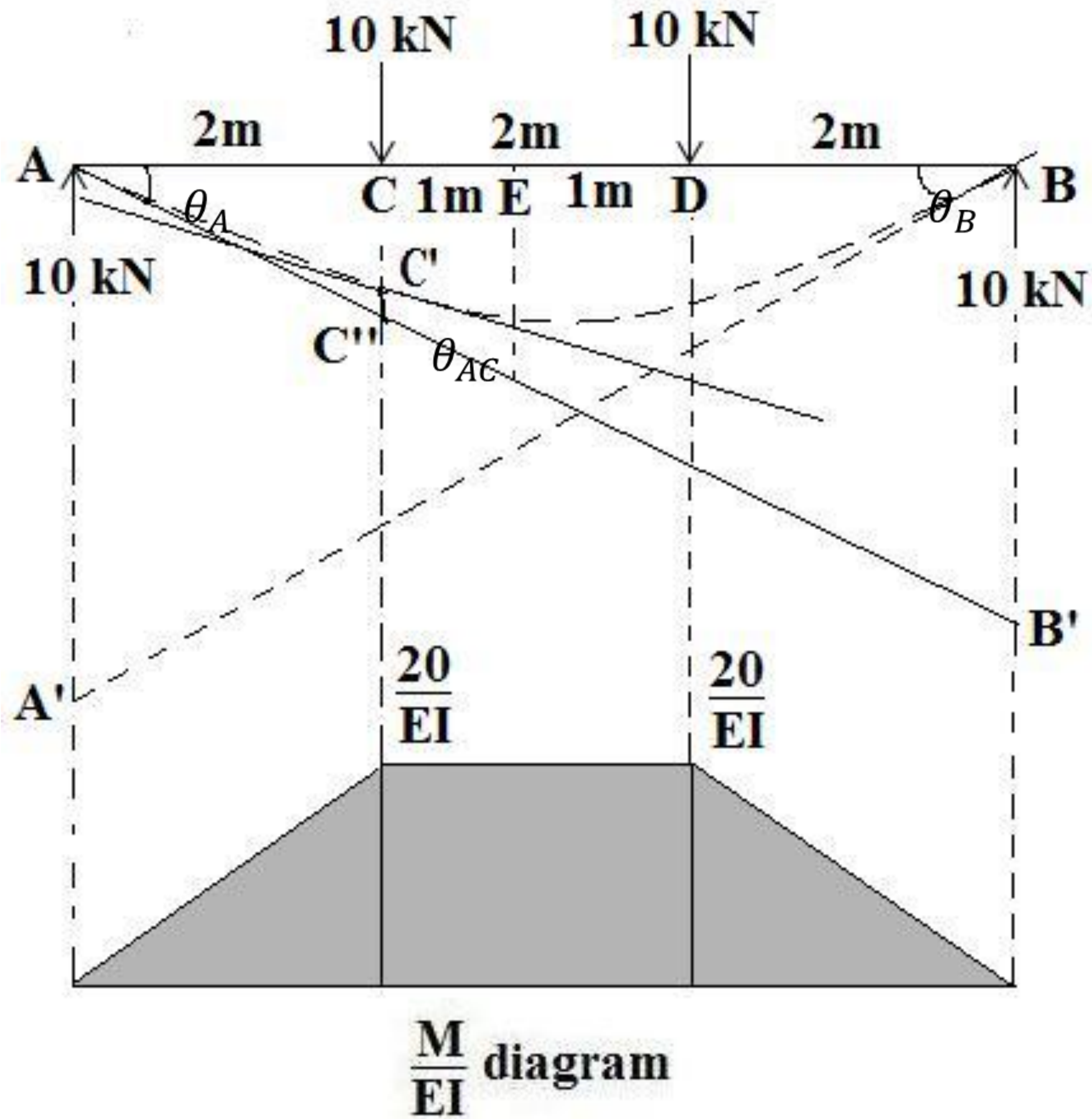
$$= \theta_A \times \frac{l}{2} - \frac{2}{3} \times \frac{l}{2} \times \frac{wl^2}{8EI} \times \frac{3l}{16}$$

$$= \frac{Wl^3}{24EI} \times \frac{l}{2} - \frac{wl^4}{128EI}$$

$$= \frac{5wl^4}{384EI}$$

Ex:79. A simply supported beam of span 6m of uniform section is subjected to two points of 10kN each at its two-third span points as shown in Fig. Determine slope at A,B,C, D & E and deflection at C &D.





10 kN
10 kN
2m
2m
2m
A
B
C 1m E 1m D
θ_A
θ_B
C'
C''
θ_{AC}
10 kN
10 kN
B'
A'
$\frac{20}{EI}$
$\frac{20}{EI}$
$\frac{M}{EI}$ diagram

Slope at A and B

BB' = Moment of area of M/EI diagram between A and B about B

$$= \frac{\frac{1}{2}\times 20\times(2+6)}{EI}\times 3$$

$$= \frac{\frac{1}{2}\times 20\times 24}{EI}$$

$$= \frac{240}{EI}$$

$$\theta_A = \frac{BB'}{l} = \frac{BB'}{6}$$

$$= \frac{40}{EI}$$

$\theta_B = \theta_A = \frac{40}{EI}$ (due to symmetry)

$Slope\ at\ C\ (\theta_{AC})$

$$= area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ C$$

$$= \frac{\frac{1}{2}\times 20\times 2}{EI} \qquad = \frac{20}{EI}$$

i.e., $\theta_{AC} = \theta_A - \theta_C = \frac{20}{EI}$

$$\therefore \theta_C = \theta_A - \frac{20}{EI} = \frac{40}{EI} - \frac{20}{EI} = \frac{20}{EI}$$

$$\therefore \theta_C = \frac{20}{EI}$$

Slope at E

$$\theta_{AE} = area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ E$$

$$= \frac{\frac{1}{2} \times 20 \times 2}{EI} + \left(1 \times \frac{20}{EI}\right) = \frac{40}{EI}$$

$$\theta_A - \theta_E = \frac{40}{EI}$$

$$\theta_E = \theta_A - \frac{40}{EI} = 0$$

Deflection at C:

$$y_C = CC' = CC'' - C'C''$$

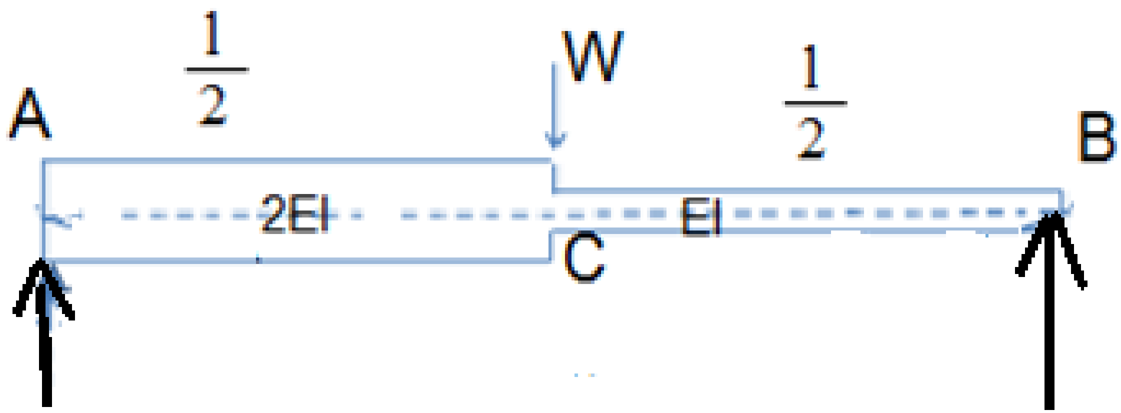
$$= \theta_A \times 2 - moment\ of\ area\ of\ \frac{M}{EI} diagram\ between\ A\ and\ C, about\ C$$

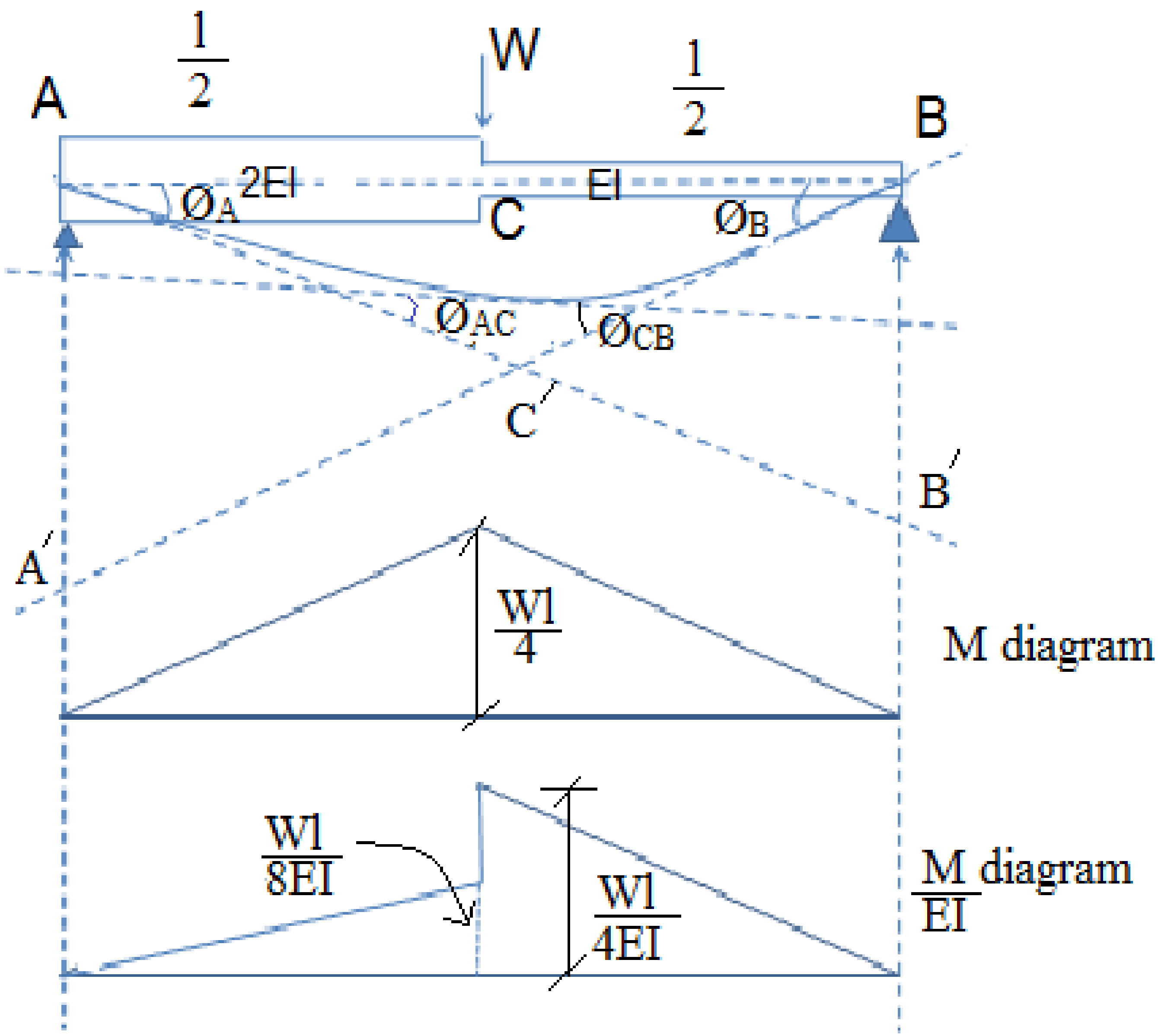
$$= \frac{40}{EI} \times 2 - \frac{\frac{1}{2} \times 20 \times 2}{EI} \times \frac{2}{3}$$

$$= \frac{80}{EI} - \frac{40}{3EI} = \frac{66.67}{EI}$$

$$\therefore y_C = \frac{66.67}{EI}$$

Ex: 80. A simply supported beam has flexural rigidity of 2EI for the left half span and EI for the right half span as shown in Fig. Determine the slope at A,B and C and deflection at C if the beam is subjected to a concentrated load of W at its mid span.





W
A
B
1
2
1
2
2EI
EI
C
ØA
ØB
ØAC
ØCB
C′
B′
A′
Wl
4
M diagram
Wl
8EI
Wl
4EI
M diagram
EI

Slope at A

By Mohr's theorem II,

$$BB' = \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{8EI} \times \left(\frac{l}{2} + \frac{l}{6}\right) + \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{4EI} \times \frac{2}{3} \times \frac{l}{2}$$

$$= \frac{Wl^3}{48EI} + \frac{Wl^3}{48EI}$$

$$= \frac{Wl^3}{24EI}$$

$$\therefore \theta_A = \frac{BB'}{l} = \frac{Wl^2}{24EI}$$

Slope at B

By Mohr's theorem II,

$$AA' = \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{4EI} \times \left(\frac{l}{2} + \frac{l}{6}\right) + \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{8EI} \times \frac{2}{3} \times \frac{l}{2}$$

$$= \frac{5Wl^3}{96EI}$$

$$\therefore \theta_B = \frac{AA'}{l} = \frac{5Wl^2}{96EI}$$

Slope at C

θ_{AC} = Moment of area of M/EI diagram

between A and C

$$= \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{8EI}$$

$$= \frac{Wl^2}{32EI}$$

We know,

$$\theta_{AC} = \theta_A - \theta_C$$

$$\therefore \theta_C = \theta_A - \theta_{AC}$$

$$= \frac{Wl^2}{24EI} - \frac{Wl^2}{32EI}$$

$$= \frac{Wl^2}{96EI}$$

(or)

$$\theta_{CB} = \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{4EI} = \frac{Wl^2}{16EI}$$

$$\theta_{CB} = \theta_C - \theta_B$$

$$\therefore \theta_C = \theta_{CB} - \theta_B$$

$$= \frac{Wl^2}{16EI} - \frac{5Wl^2}{96EI}$$

$$= \frac{-Wl^2}{96EI}$$

Deflection at C

C′C″ = Moment of area of M/EI diagram between A and C about C

$$= \frac{1}{2} \times \frac{l}{2} \times \frac{Wl}{8EI} \times \frac{l}{6}$$

$$= \frac{Wl^3}{192EI}$$

$$y_C = CC' = CC'' - C'C''$$

$$= \theta_A \frac{l}{2} - C'C''$$

$$= \frac{Wl^2}{24EI} \times \frac{l}{2} - \frac{Wl^3}{192EI}$$

$$= \frac{3Wl^3}{192EI}$$

$$= \frac{Wl^3}{64EI} \; downward$$

Ex.81 The middle half of the beam shown in Fig. has a moment of inertia 1.5 times that of the rest of the beam. Find the mid span deflection.

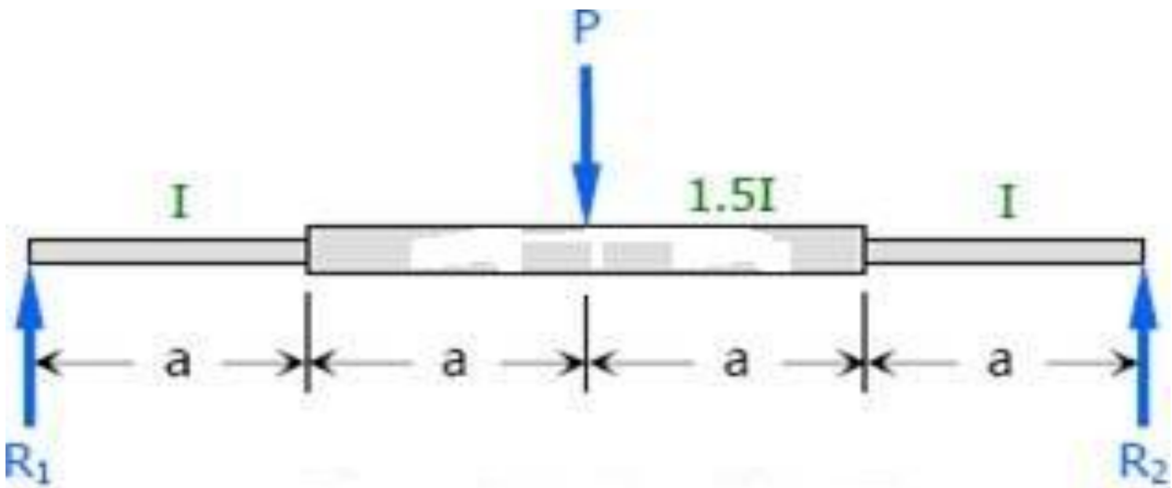
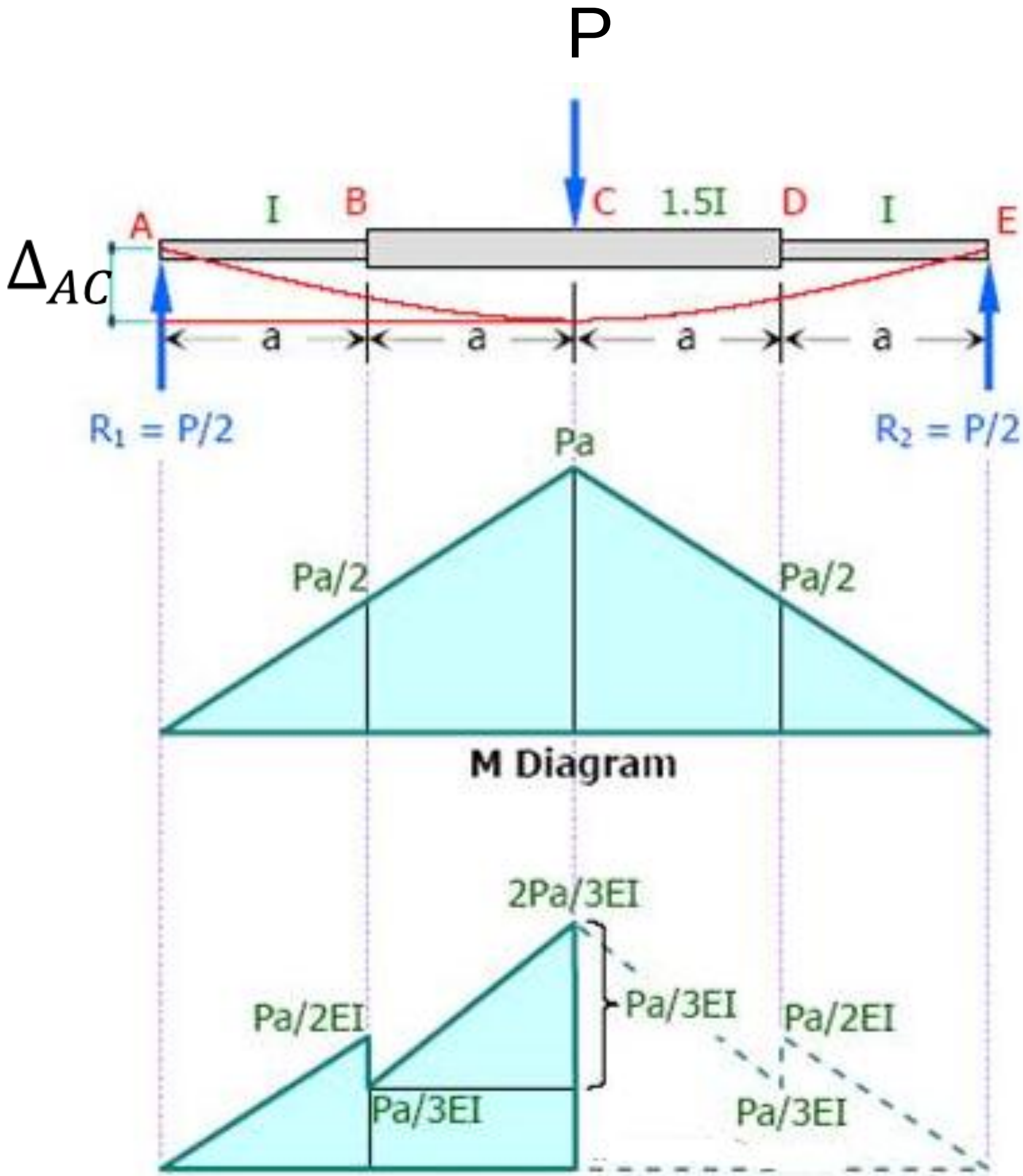


Fig.



$\frac{M}{EI}$ *DIAGRAM*

$$\Delta_{AC} = \frac{1}{EI}(Area_{AC})\bar{X}_A$$

$$\Delta_{AC} = \frac{1}{2}a\left(\frac{Pa}{2EI}\right)\left(\frac{2}{3}a\right) + a\left(\frac{Pa}{3EI}\right)\left(\frac{3}{2}a\right)$$

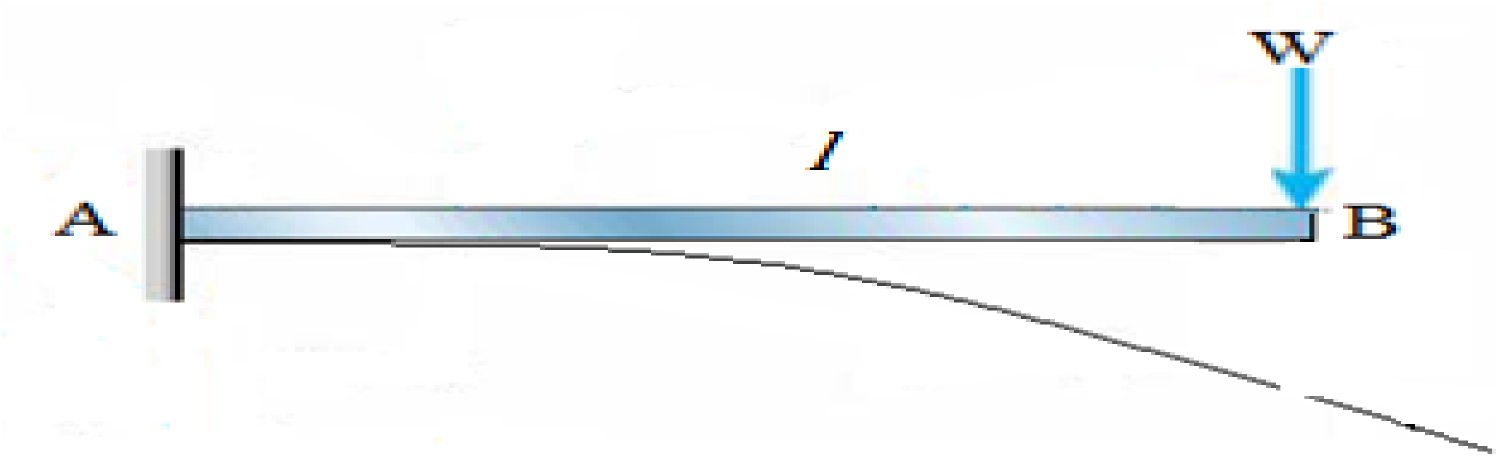
$$+\frac{1}{2}a\left(\frac{2Pa}{3EI} - \frac{Pa}{3EI}\right)(\frac{5}{3}a)$$

$$\Delta_{AC} = \frac{Pa^3}{6EI} + \frac{Pa^3}{2EI} + \frac{5Pa^3}{18EI}$$

$$\Delta_{AC} = \frac{17Pa^3}{18EI}$$

$$\delta_{mid\ span} = \frac{17Pa^3}{18EI}$$

<u>Ex:9.7</u> A cantilever subjected to a point load W at the free end. Find the slope and deflection at the free end.



$$\theta_{AB} = area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ B$$

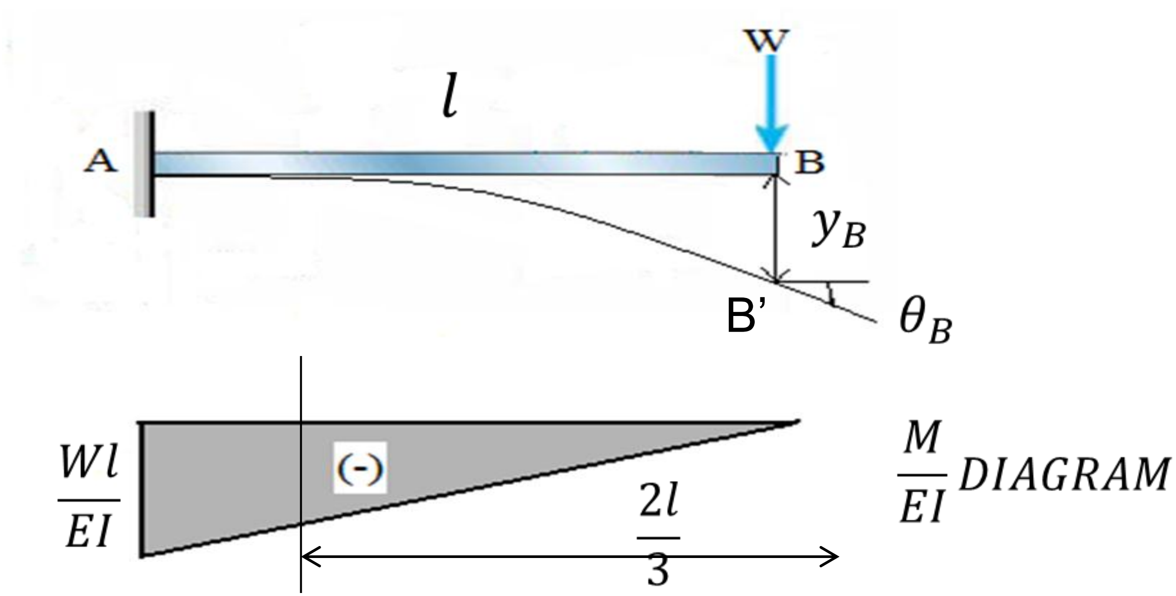
$$= \frac{1}{2} \times l \times \frac{-Wl}{EI} = \frac{-Wl^2}{2EI}$$

$$\theta_A - \theta_B = \frac{-Wl^2}{2EI}$$

$$\theta_A = 0$$

$$\therefore \theta_B = \frac{Wl^2}{2EI}$$

To find y_B:



y_B = BB' = Moment of area of M/EI

diagram between A and B, about B

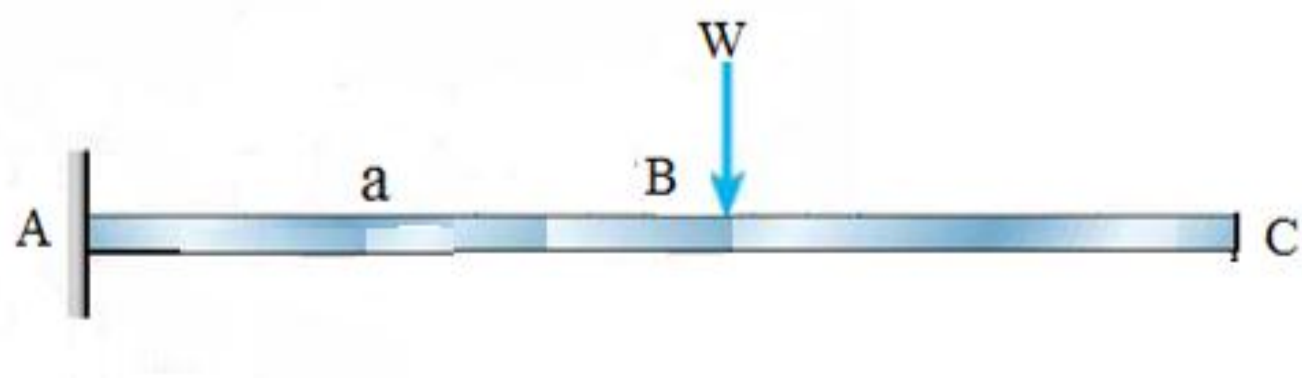
$$= \left(\frac{-Wl^2}{2EI}\right)\left(\frac{2l}{3}\right)$$

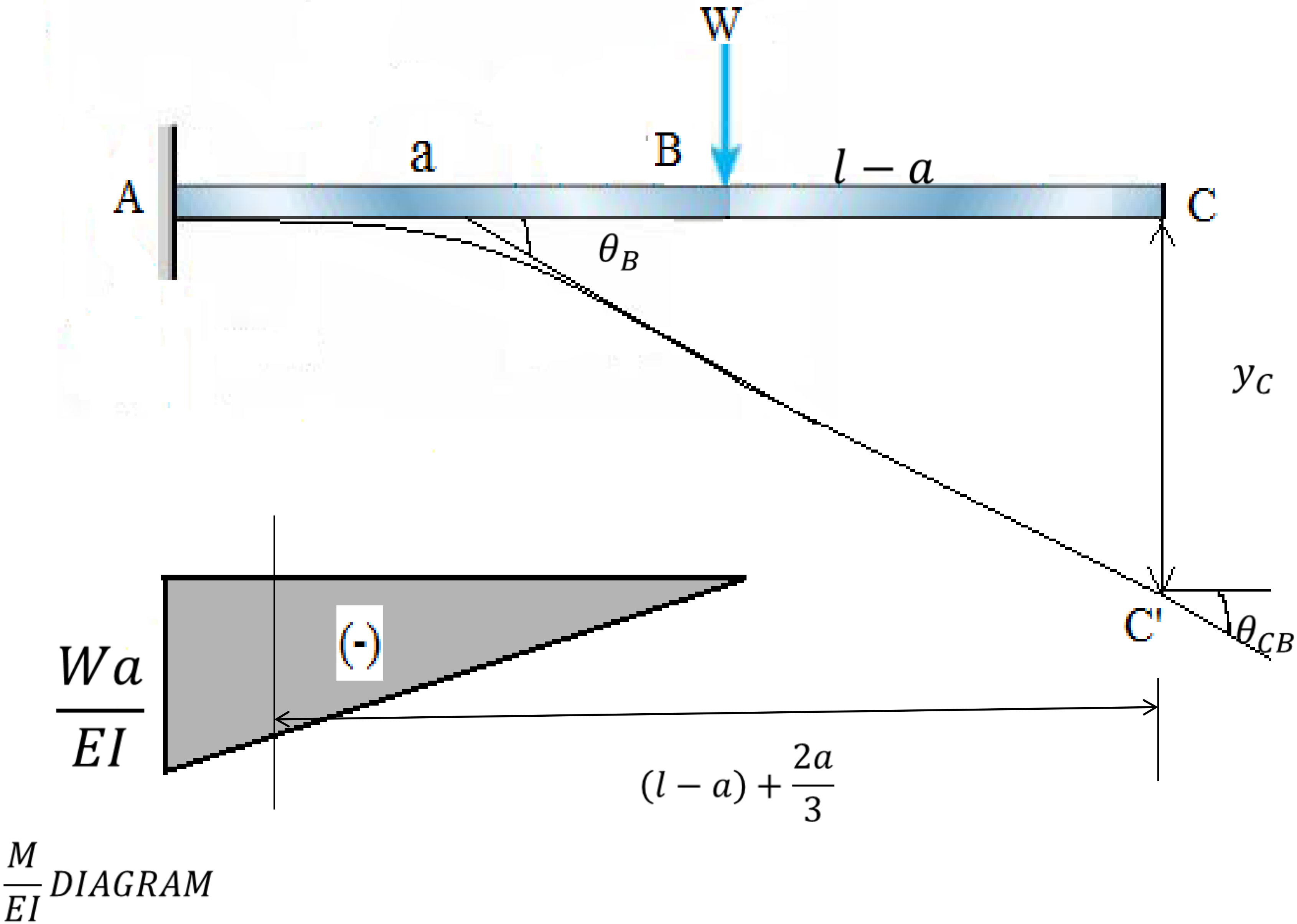
$$= \frac{-Wl^3}{3EI}$$

-ve sign ⇒ intercept is above deflection curve

$$\therefore\ y_B = \frac{Wl^3}{3EI}\ (downward)$$

Ex:83. A cantilever subjected to a point load W applied at a distance “a” from the fixed end. Find the slope and deflection at the free end.




W
a
B
$l-a$
A
C
θ_B
y_C
C'
θ_{CB}
(-)
$\frac{Wa}{EI}$
$(l-a)+\frac{2a}{3}$
$\frac{M}{EI}$ DIAGRAM

Slope at B and C

Change in slope between A and B

$$\theta_{AB} = area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ B$$

$$= \frac{1}{2} \times a \times \frac{-Wa}{EI}$$

$$= \frac{-Wa^2}{2EI}$$

$$\theta_{AB} = \theta_A - \theta_B = \frac{-Wa^2}{2EI}$$

$\therefore\ \theta_B = \frac{Wa^2}{2EI}$(clockwise)

Change in slope between B & C,

$$\theta_{CB} = area\ of\ \frac{M}{EI}\ diagram\ between\ B\ and\ C$$

$$= 0$$

$$\theta_C - \theta_B = 0$$

$$\therefore \theta_C = \theta_B = \frac{Wa^2}{2EI}$$

Deflection at B

$y_B =$ BB' = Moment of area of M/EI

diagram between A and B, about B

$$= \left(\frac{-Wa^2}{2EI}\right)\left(\frac{2a}{3}\right)$$

$$= \frac{-Wa^3}{3EI}$$

-ve sign ⇒ intercept is above deflection curve

$$\therefore\ y_B = \frac{Wa^3}{3EI}\ downward$$

Deflection at C

$y_C = \mathrm{CC'}$ = Moment of area of M/EI

diagram between A and C, about C

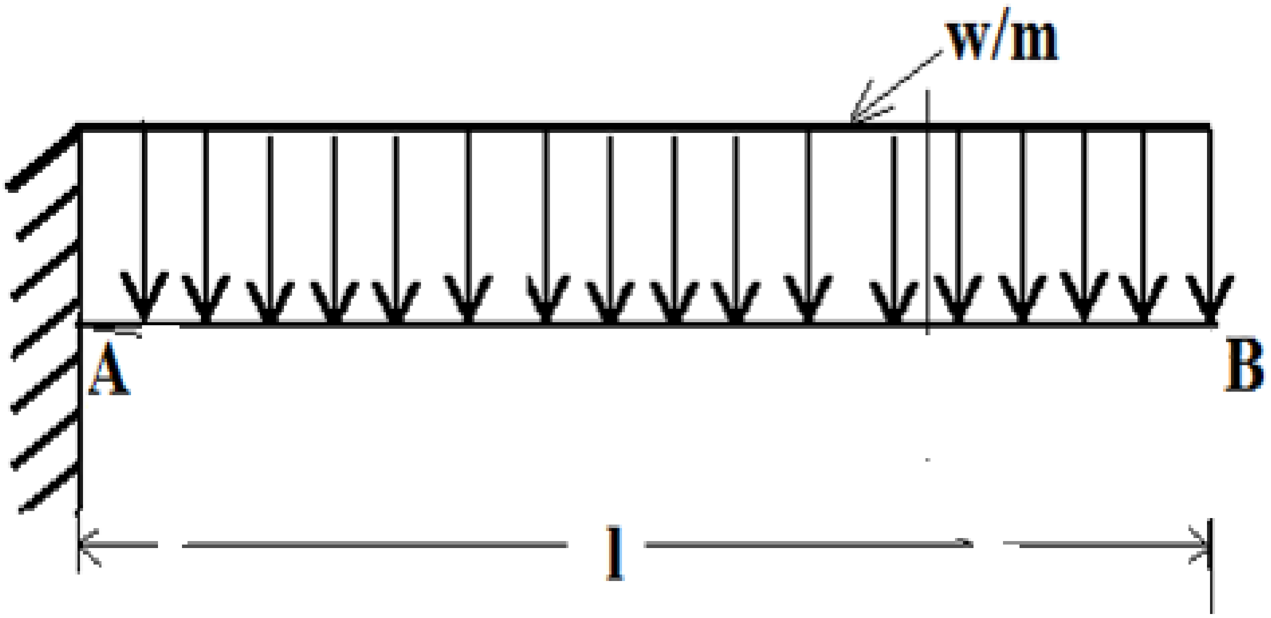
$$= \left(\frac{-Wa^2}{2EI}\right)\left(l - a + \frac{2a}{3}\right)$$

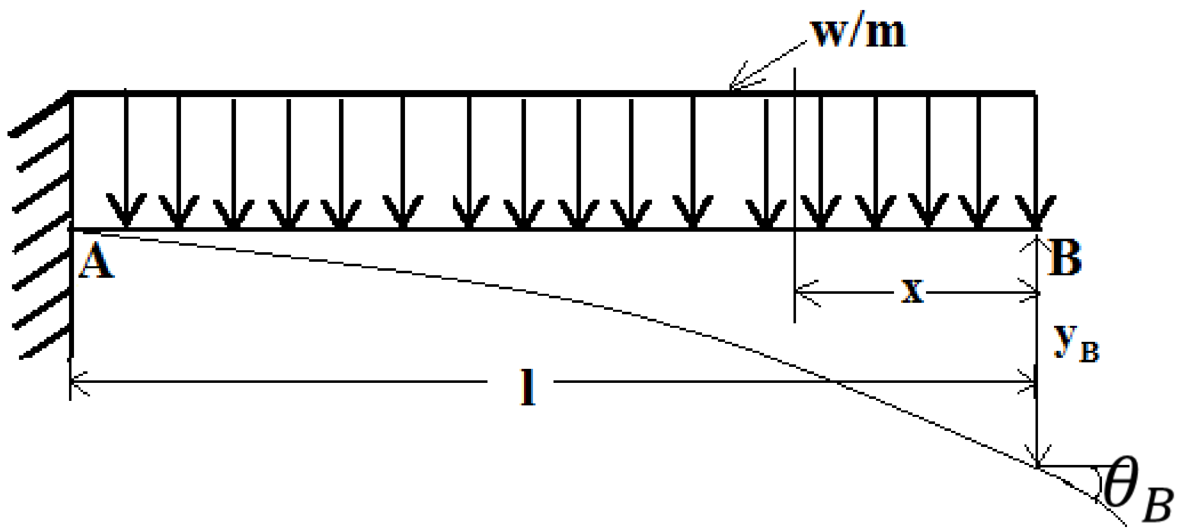
$$= \frac{-Wa^2}{2EI}\left(l - \frac{a}{3}\right)$$

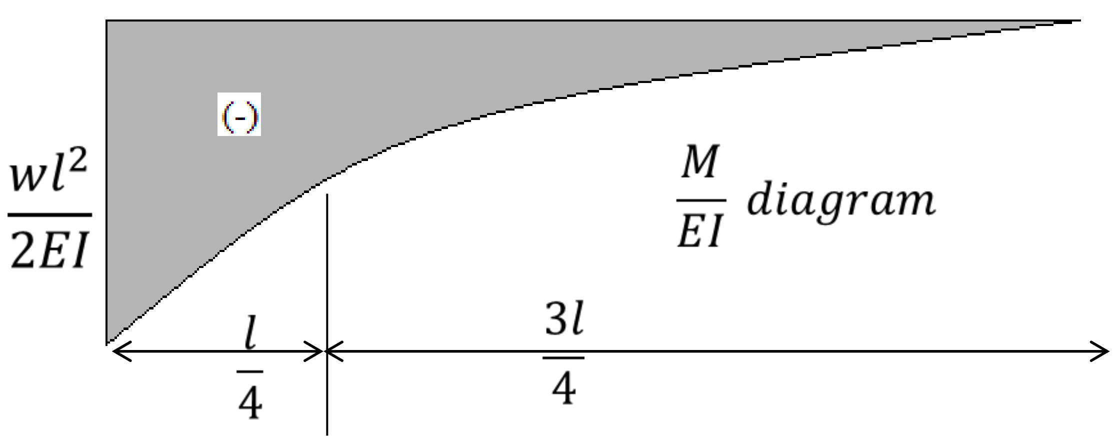
-ve sign ⇒ intercept is above deflection curve

$$\therefore\ y_C = \frac{Wa^2}{2EI}\left(l - \frac{a}{3}\right) downward$$

Ex: 84. A cantilever subjected to a U.D.L over the entire length. Find the slope and deflection at the free end.




w/m
A
B
x
y_B
l
θ_B


(-)
$\frac{wl^2}{2EI}$
$\frac{M}{EI}$ diagram
$\frac{l}{4}$
$\frac{3l}{4}$

Slope at B

$$chnge\ in\ slope\ between\ A\ and\ B, \theta_{AB}$$

$$= area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ B$$

$$= \frac{1}{3} \times l \times (\frac{-Wl^2}{2EI})$$

$$= \frac{-Wl^3}{6EI}$$

$$\theta_{AB} = \theta_A - \theta_B = \frac{-Wl^3}{6EI}$$

We know, $\theta_A = 0$ $\quad\quad \therefore \theta_B = \frac{Wl^3}{6EI} (clockwise)$

Deflection at B

$y_B =$ BB' = Moment of area of M/EI

diagram between A and B, about B

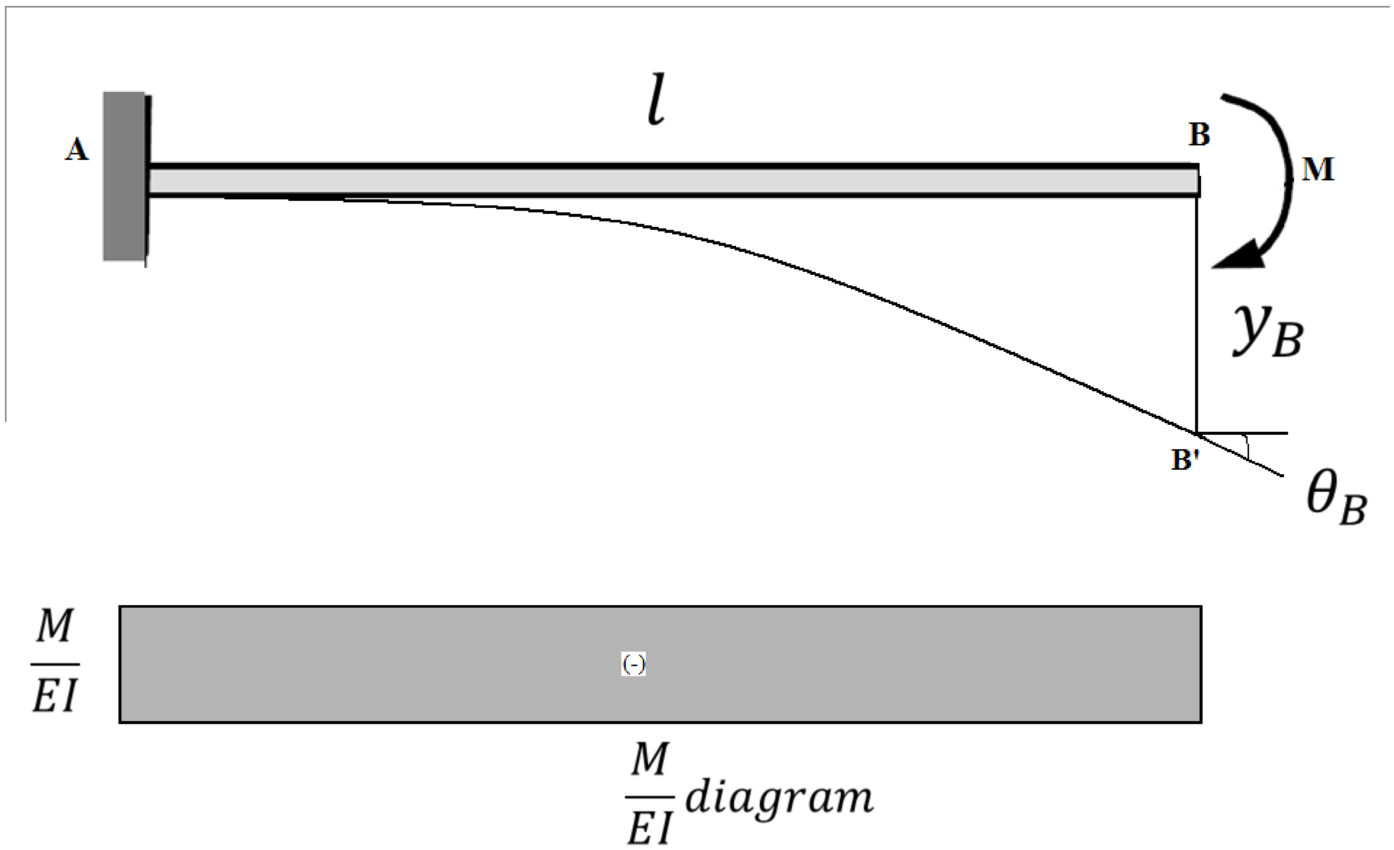
$$= \left(\frac{1}{3} \times l \times \frac{-Wl^2}{2EI}\right) \times \frac{3}{4} l$$

$$= \frac{-Wl^4}{8EI}$$

-ve sign ⇒ intercept is above deflection curve

$$\therefore y_B = \frac{Wl^4}{8EI}\ downward$$

Ex: 85. A cantilever is subjected to a couple at free end . Find the slope and deflection at the free end.



Slope at free end B

$$\theta_{AB} = area\ of\ \frac{M}{EI}\ diagram\ between\ A\ and\ B$$

$$= \frac{-M}{EI} \times l$$

$$= \frac{-Ml}{EI}$$

$$\theta_{AB} = \theta_A - \theta_B = \frac{-Ml}{EI}$$

$$\theta_B = \frac{Ml}{EI} (clockwise)$$

Deflection at free end

y_B = BB' = Moment of area of M/EI

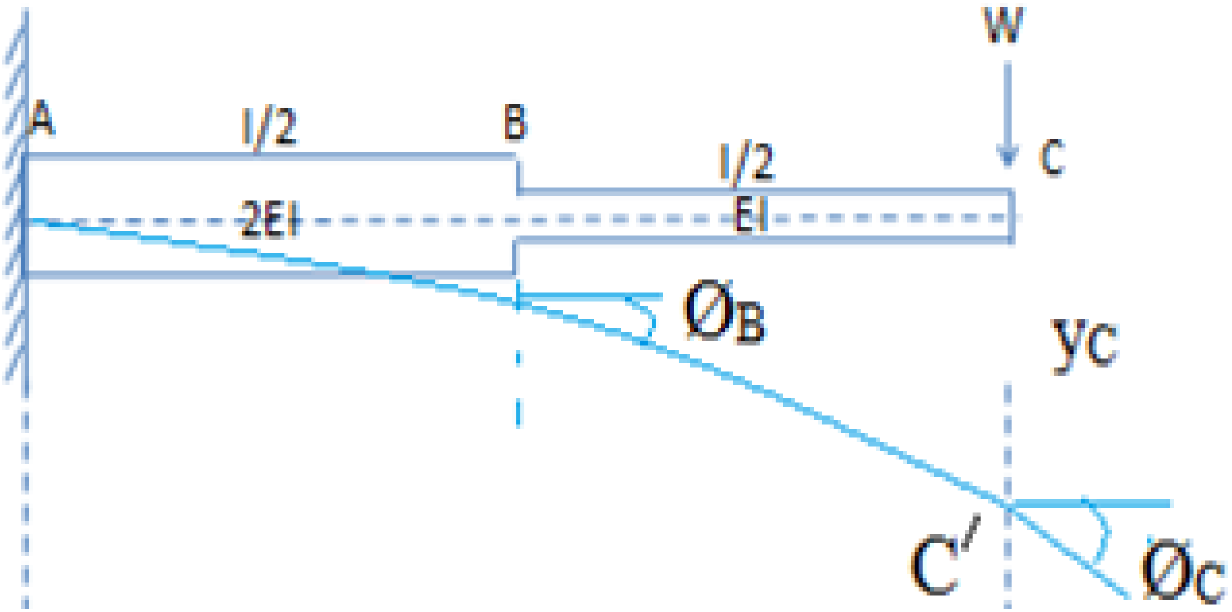
diagram between A and B, about B

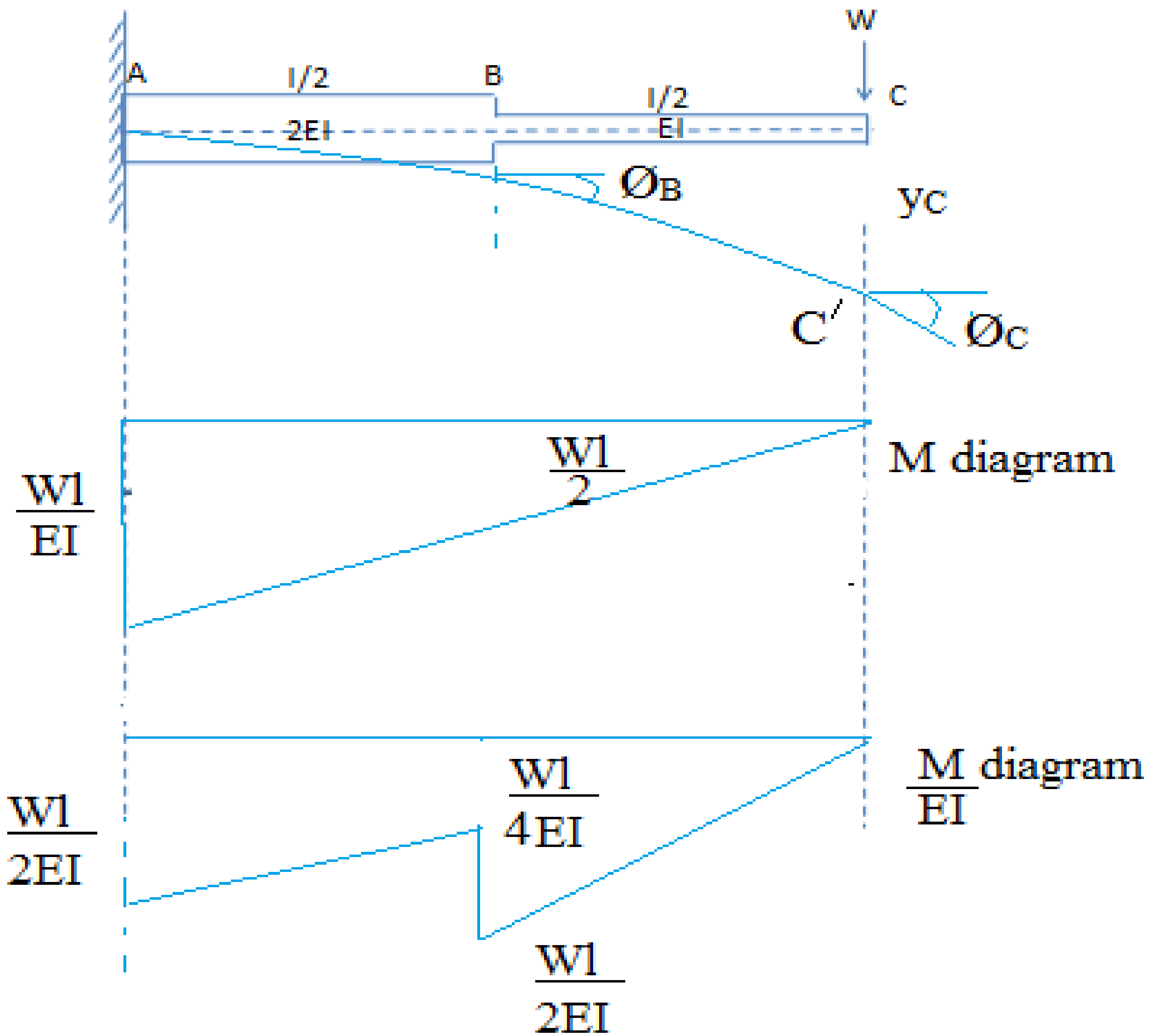
$$= \left(\frac{-Ml}{EI}\right) \times \frac{l}{2}$$

$$= \frac{-Ml^2}{2EI}$$

$$\therefore y_B = \frac{Ml^2}{2EI}$$

Ex:86. A cantilever of span ‘l’ has flexural rigidity of 2EI for a half span from fixed end and EI for the remaining half span as shown in Fig. Determine the slope and deflection at B and C.





W
A
l/2
B
l/2
C
2EI
EI
ØB
yc
C′
ØC
Wl
2
M diagram
Wl
EI
Wl
4EI
M
EI
diagram
Wl
2EI
Wl
2EI

Slope at B

$$\theta_{AB} = -\frac{1}{2} \times \frac{l}{2} \times \left(\frac{3Wl}{4EI}\right)$$

$$= -\frac{3Wl^2}{16EI}$$

We know,

$$\theta_{AB} = \theta_A - \theta_B$$

$$= 0 - \theta_B$$

$$\therefore \theta_B = \frac{3Wl^2}{16EI} \ clockwise$$

Deflection at B

$$y_B = BB' = \frac{-3Wl^2}{16EI} \times \frac{\frac{l}{6}\left(2\frac{Wl}{2EI}+\frac{Wl}{4EI}\right)}{\left(\frac{Wl}{4EI}+\frac{Wl}{2EI}\right)}$$

$$= \frac{-3Wl^2}{16EI} \times \frac{5l}{18}$$

$$= \frac{-5Wl^3}{96EI}$$

(-ve sign indicates the intercept is above the elastic curve)

Slope at C

$$\theta_{AC} = \frac{-3Wl^2}{16EI} + \frac{1}{2} \times \frac{l}{2} \times \left(\frac{-Wl}{2EI}\right)$$

$$= \frac{-3Wl^2}{16EI} + \left(\frac{-Wl^2}{8EI}\right)$$

$$= \frac{-5Wl^2}{16EI}$$

$$\theta_{AC} = \theta_A - \theta_C = -\theta_C$$

$$\therefore \theta_C = \frac{5Wl^2}{16EI}\ clockwise$$

Deflection at C

$$y_C = CC' = \frac{-3Wl^2}{16EI}\left(\frac{l}{2} + \frac{5l}{18}\right) + \left(-\frac{Wl^2}{8EI}\right) \times \frac{l}{3}$$

$$= \frac{-3Wl^3}{16EI}\left(\frac{14}{18}\right) - \frac{Wl^3}{24EI}$$

$$y_C = \frac{-7Wl^3}{48EI} - \frac{Wl^3}{24EI}$$

$$= -\frac{3Wl^3}{16EI}$$

$$\therefore y_C = -\frac{3Wl^3}{16EI}$$

(-ve sign indicates the intercept is above the deflection curve)

Ex.9.12. Find the maximum deflection for the cantilever beam loaded as shown in Fig.9.26. if the cross section is 50 mm wide by 150 mm high. Use E = 69 GPa.

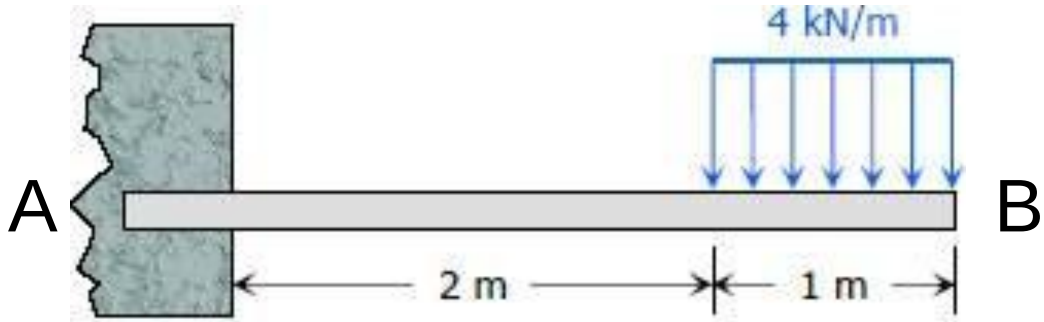
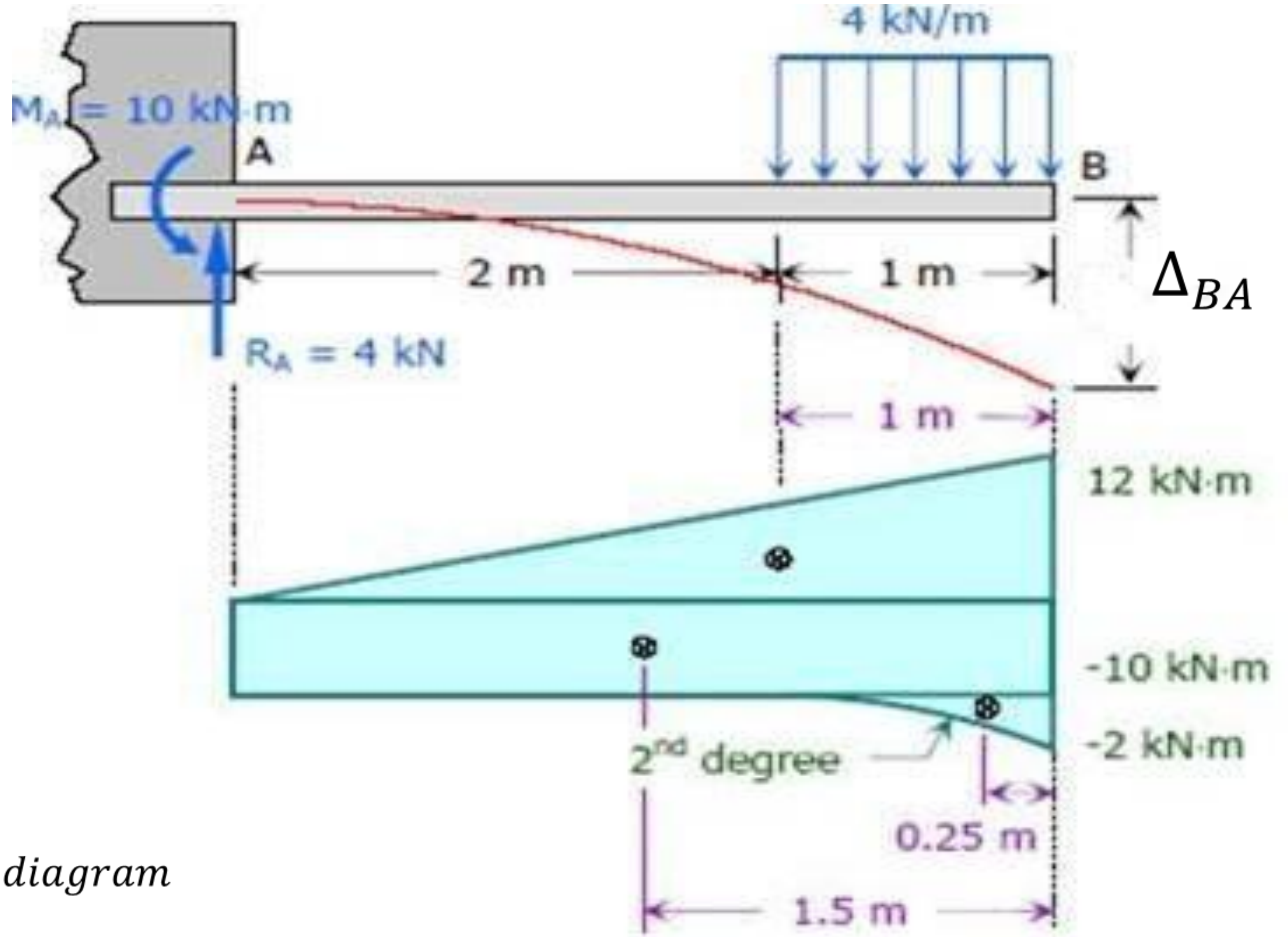


Fig.9.26

$$R_A = 4(1) = 4kN$$
$$M_A = 4(1)(2.5) = 10kN.m$$



4 kN/m
M_A = 10 kN·m
A
B
2 m
1 m
Δ_{BA}
R_A = 4 kN
1 m
12 kN·m
-10 kN·m
-2 kN·m
2nd degree
0.25 m
1.5 m

M $diagram$

$$\Delta_{BA} = \frac{1}{EI}(Area_{AB})\bar{X}_B$$

$$\Delta_{BA} = \frac{1}{69000\left[\frac{50(150)^3}{12}\right]} \times \left\{\frac{1}{2}(3)(12)(1) - 3(10)(1.5) - \frac{1}{3}(1)(2)(0.25)\right\}(1000)^4$$

$$\Delta_{BA} = -28\; mm$$

$$\therefore \qquad \delta_{max} = 28\; mm \qquad \textbf{\textit{(answer)}}$$

Ex.9.13. Find the maximum value of EIδ for the beam shown in Fig.9.28

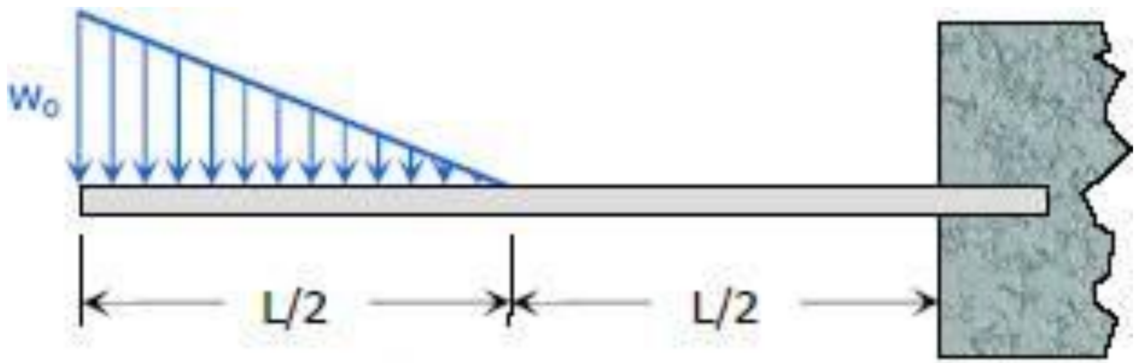
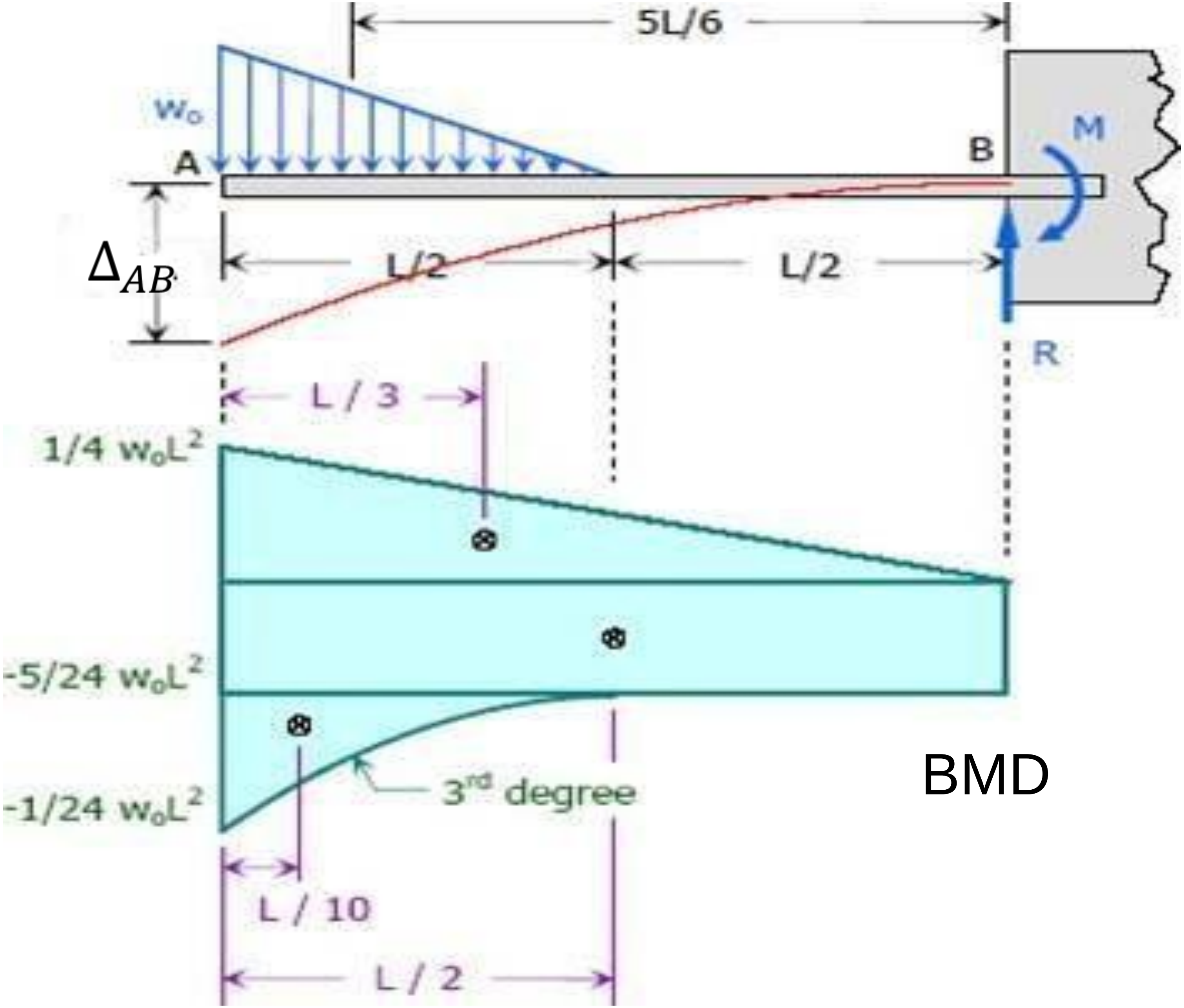


Fig.9.28

$$R = \frac{1}{2}\left(\frac{1}{2}L\right)(w_0) = \frac{1}{4}w_0\mathrm{L}$$

$$M = \frac{1}{2}\left(\frac{1}{2}L\right)(w_0)\left(\frac{5}{6}L\right) = \frac{5}{24}w_0L^2$$


5L/6
w_o
A
B
M
Δ_{AB}
L/2
L/2
R
L / 3
1/4 w_oL^2
-5/24 w_oL^2
-1/24 w_oL^2
3rd degree
L / 10
L / 2

BMD

$$EI\Delta_{AB} = (Area_{AB})\bar{X}_A$$

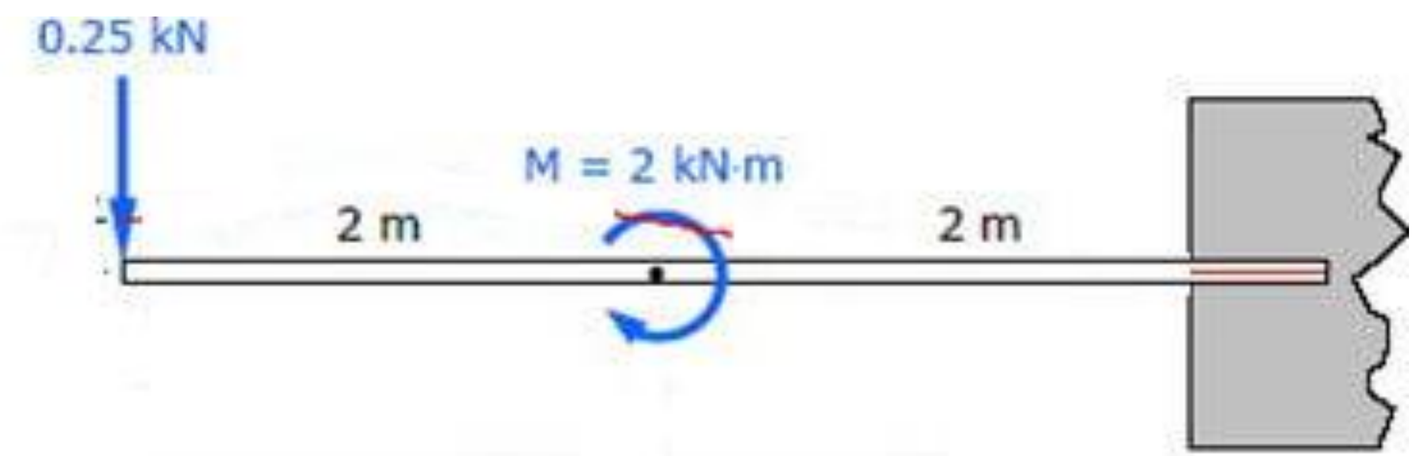
$$= \frac{1}{2}L\left(\frac{1}{4}w_0L^2\right)\left(\frac{1}{3}L\right) - \left(L \times \frac{5}{24}w_0L^2\right)\left(\frac{1}{2}L\right) -$$

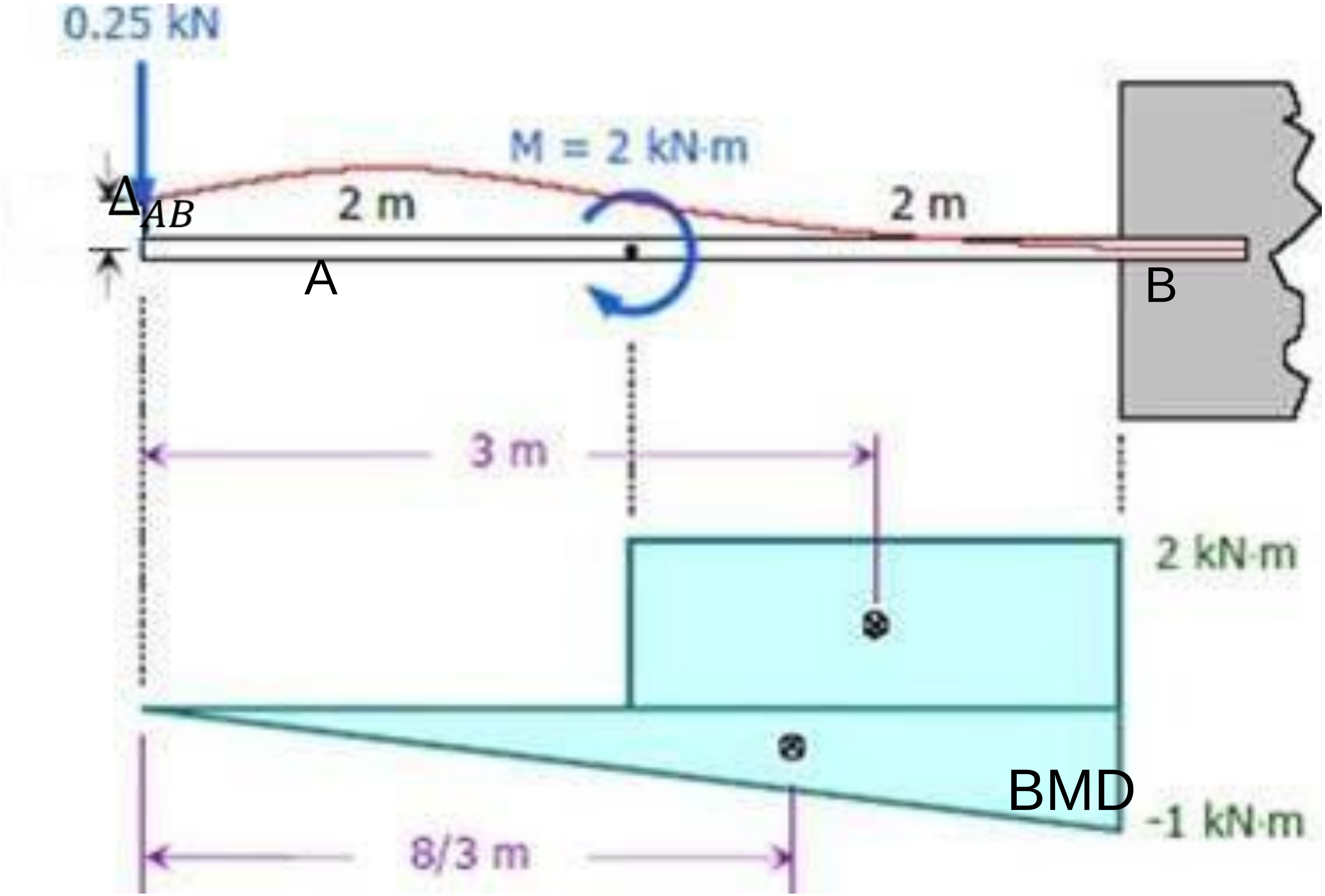
$$\frac{1}{4}(\frac{1}{2}L)(\frac{1}{24}w_0L^2)(\frac{1}{10}L)$$

$$= \frac{1}{24}w_0L^4 - \frac{5}{48}w_0L^4 - \frac{1}{1920}w_0L^4 = -\frac{121}{1920}w_0L^4$$

$$EI\delta_{max} = \frac{121}{1920}w_0L^4$$

Ex.89. For the cantilever beam shown in Fig. determine the value of EIδ at the left end. Is this deflection upward or downward?




0.25 kN
M = 2 kN·m
Δ_{AB}
2 m
2 m
A
B
3 m
2 kN·m
BMD
-1 kN·m
8/3 m

$$\Delta_{AB} = \frac{1}{EI}(Area_{AB})\bar{X}_A$$

$$EI\Delta_{AB} = 2(2)(3) - \frac{1}{2}(4)(1)(\frac{8}{3})$$

$$EI\Delta_{AB} = \frac{20}{3} = 6.67\ kN.m^3$$

$\therefore$ $EI\delta = 6.67\ kN{\cdot}m^3$ upward (***answer)***

THEORY OF COLUMNS

We come across various instances of members subjected to compressive loads.

These members are given different names depending upon the particular situation in which they are placed.

- Post is a general term applied to a compression member.
- Columns, pillars and stanchions are vertical members used in building frames.
- Strut is a compression member in a truss

  (Tie is a tension member in a truss)

  Other examples: piston rods, connecting rods, side links in forging machines etc.,
- Boom is the principal compression member in a crane.

A structural member whose lateral dimensions are small as compared to its length and subjected to compressive force is known as a strut.

A strut may be horizontal, inclined or vertical. But a vertical strut, used in buildings or frames is called column.

CLASSIFICATION OF COLUMNS

1. Short column:

Short column fails by crushing (compressive yielding) of the material.

2. Long column:

Long column fails by buckling or bending.

(geometric or configuration failure)

3. Intermediate column:

Intermediate column fails by combined buckling and crushing. (failure due to both material crushing and geometrical instability)

BUCKLING

- When a slender member is subjected to an axial compressive load, it may fail by a condition called ***buckling***.
- Buckling is a geometric instability in which the lateral displacement of the axial member can suddenly become very large (*Figure 10.1*).

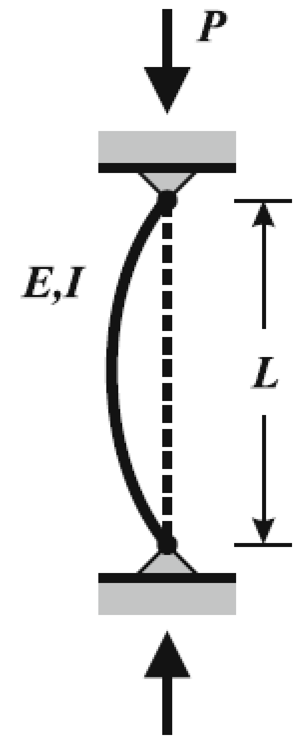


Figure 10.1. Original shape (dashed) and buckled shape (solid) of a pinned–pinned column under compressive load.

• Buckling:

Examples of structural members and systems that are subjected to loads that may cause buckling are:

1. Building columns that transfer loads to the ground;
2. Truss members in compression;
3. Machine elements
4. Submarine hulls subjected to water pressure

STATES OF EQUILIBRIUM

(concept of elastic stability)

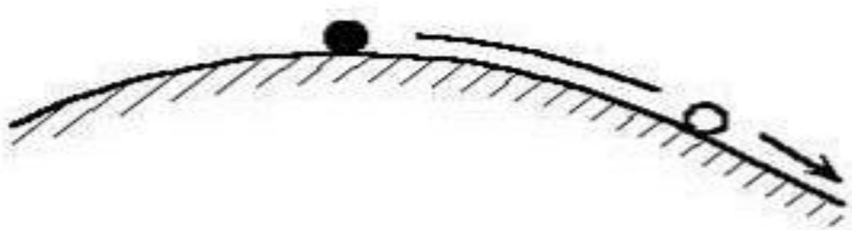
From mechanics it is known that a body may be in three types of equilibrium

i.e., Stable equilibrium

Neutral equilibrium

Unstable equilibrium

(a) Stable



(b) Unstable

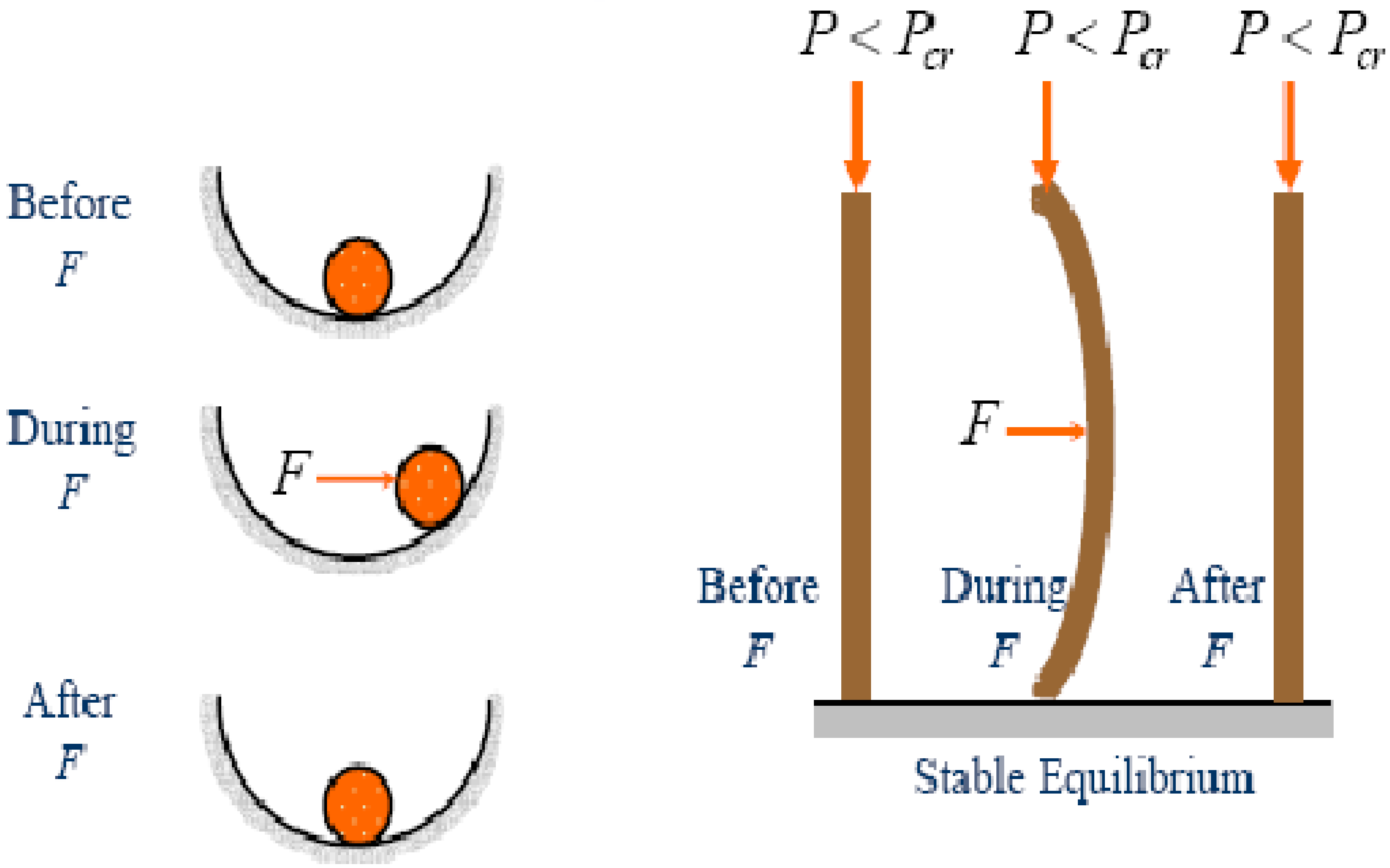
(c) Neutral

Fig. 10.3 The three states of equilibrium

Mechanism of buckling

- Let us consider Figures 10.4 , 10.5 , and 10. 6 and study them very carefully
- In Figure 10.4 some axial load P is applied to the column,
- The column is then given a small deflection by applying the small lateral force F
- If the load P is sufficiently small, when the force F is removed, the column will go back to its original straight condition

Mechanism of Buckling



Mechanism of Buckling

- The column will go back to its original straight condition just as the ball returns to the bottom of the curved container.
- In Fig.10.4 of the ball and the curved container, gravity tends to restore the ball to its original position, while for the column the elasticity of the column itself acts as restoring force
- This action constitutes stable equilibrium

Mechanism of Buckling

- The same procedure can be repeated for increased value of the load P until some critical value P_{cr} is reached, as shown in Figure 10.5.
- When the column carries this load, and a lateral force F is applied and removed, the column will remain in the slightly deflection position.
- The elastic restoring force of the column is not sufficient to return the column to its original straight position but is sufficient to prevent excessive deflection of the column

– Mechanism of Buckling

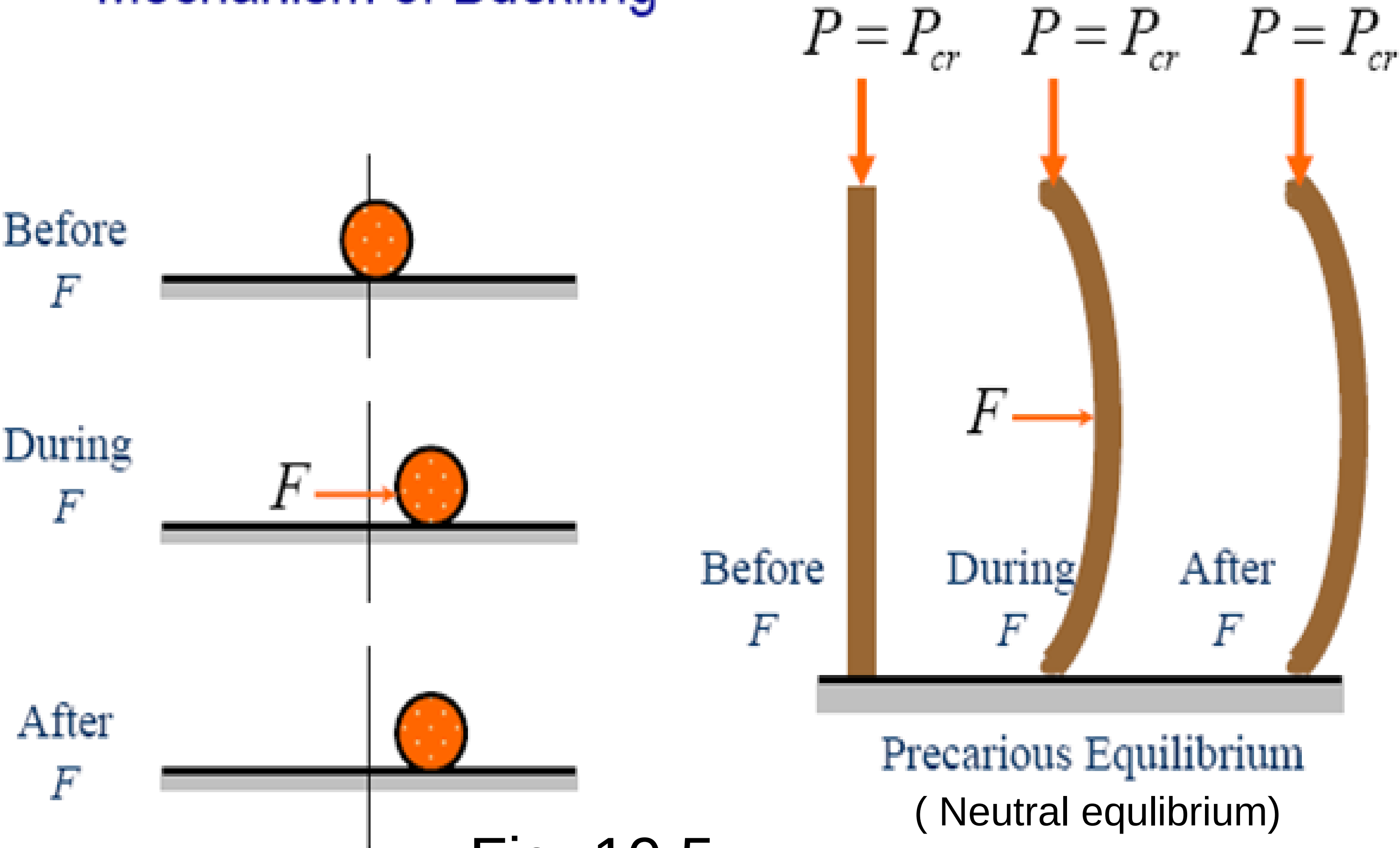


Fig. 10.5

Mechanism of Buckling

- In figure 10.5 of the ball and the flat surface, the amount of deflection will depend on the magnitude of the lateral force F.
- Hence, the column can be in equilibrium in an infinite number of slightly bent positions.
- The action constitutes <u>neutral or precarious equilibrium</u>

Mechanism of buckling

- If the column is subjected to an axial compressive load P that exceeds P_{cr} as shown in Figure 10.6 and a lateral force F is applied and removed, the column will bend considerably.
- That is, the elastic restoring force of the column is insufficient to prevent a small disturbance from growing into an excessively large deflection.

– Mechanism of Buckling

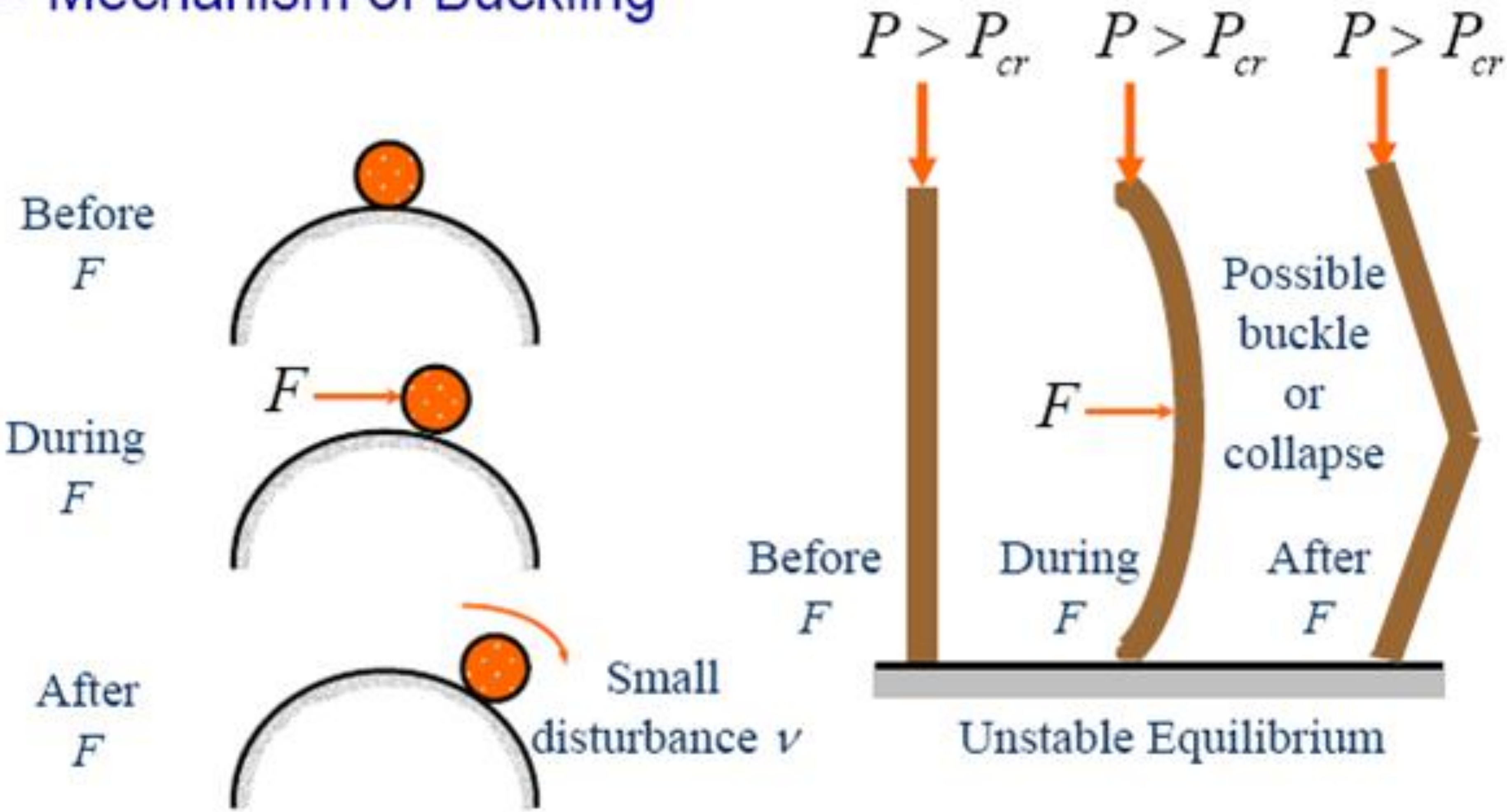


Fig. 10.6

Mechanism of buckling

- Depending on the magnitude of P, the column either will remain in the bent position or will completely collapse and fracture, just as the ball will roll off the curved surface as in Figure 10.6.
- This type of behavior indicates that for axial loads greater than straight position of a column is one of unstable equilibrium in that a small disturbance will tend to grow into an excessive deformation.

Buckling

Definition

“ Buckling can be defined as the sudden large deformation of structure due to a slight increase in an existing load under which the structure had exhibited little, if any, deformation before the load was increased ”

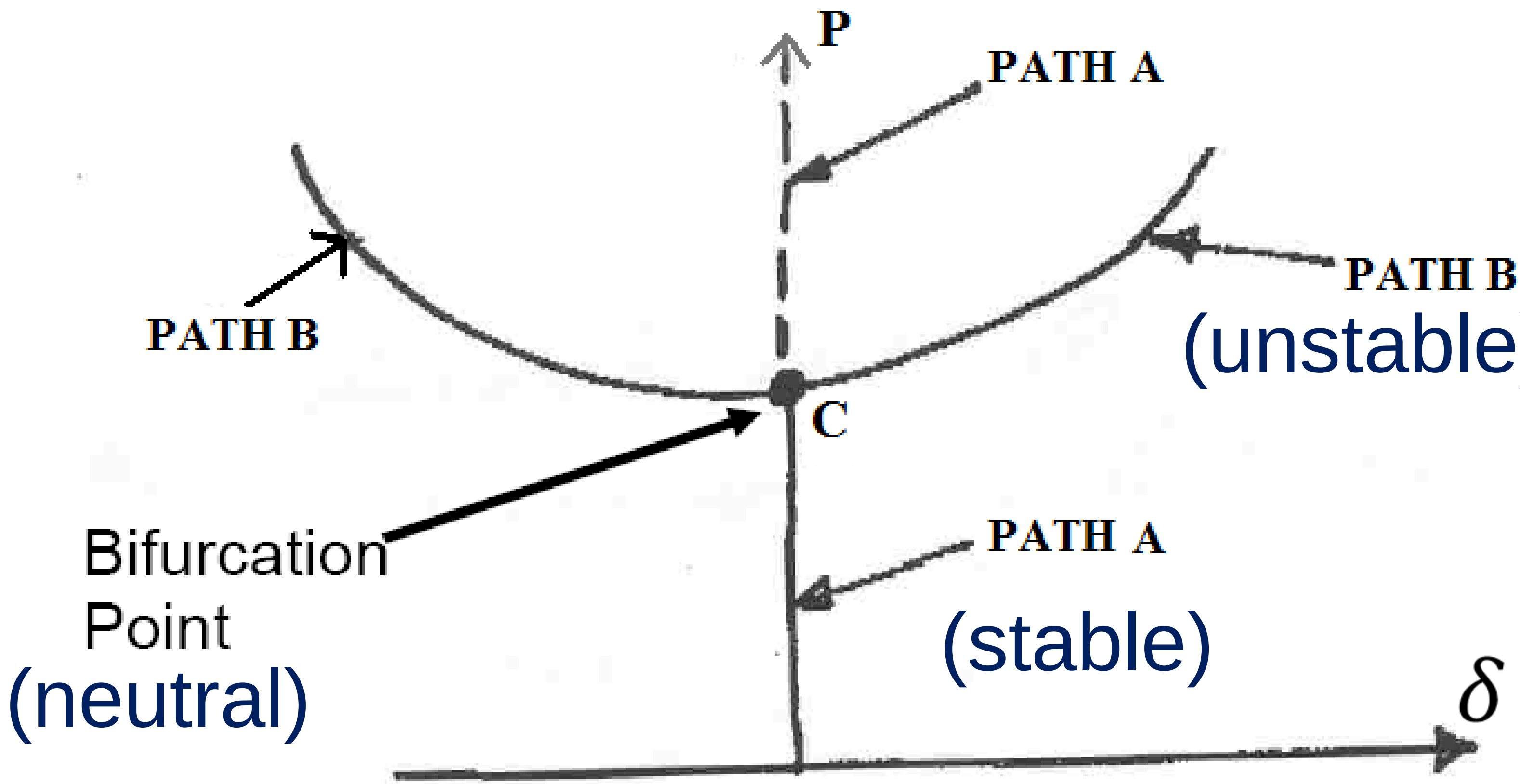


Figure 10.7 : Equilibrium Paths.

In this figure 10.7, the two solutions are identified as branching paths A and B.

Path A (the trivial solution) is shown as the <u>fundamental path</u>.

Path B on the other than is shown as the <u>post-buckling path</u>.

The point C where the fundamental path meets the post-buckling path is shown as a bifurcation point (a point where the fundamental path bifurcate (splits) into two or more paths).

The fundamental path is stable for each point below C and unstable for each point above C.

The load corresponding to the bifurcation point (neutral equilibrium) is the critical load.

i.e., P_{cr} is the critical load (the point at which we lose stability on the fundamental path).

EULER'S COLUMN THEORY

Euler derived an equation, for the buckling load of long column based on bending stress (neglecting the effect of direct stress).

This may be justified with the statement, that the direct stress induced in a long column is negligible as compared to the bending stress.

Therefore this cannot be used in short column.

Assumptions in the Euler's Column Theory

1. Initially the column is perfectly straight and the load applied is truly axial.

2. The cross section of the column is uniform throughout its length.

3. The column material is perfectly elastic, homogeneous and isotropic and thus obeys Hooke's law.

4. The length of column is very large as compared to its cross-sectional dimensions.

5. The shortening of column, due to direct compression (being very small) is neglected.

6. The failure of column occurs due to buckling alone

7. The self weight of the column is neglected

8. The ends of the column are frictionless

9. The direct stress is very small compared with the bending stress corresponding to the buckling conditions.

Basic differential equation for buckling

$$EIy'' = M_x$$

where **y** is the lateral deflection of the column at any distance **x** from the origin.

E is the modulus of elasticity of the column material,

I represent the second moment of area (moment of inertia of the cross sectional area) about the bending axis(weaker axis), and

$\boldsymbol{M_x}$ represents the bending moment at a distance x from origin.

Sign Conventions

Fig. 10.7

1. A moment, which tends to bend the column with convexity towards its initial central line as shown in Fig. 10.7(a) is taken as positive.
2. A moment, which tends to bend the column with its concavity towards its initial central line as shown in Fig. 10.7 (b) is taken as negative.

END CONDITIONS

There can be three important end conditions at the end of the column

(i) Fixed end: In this case, the end is fixed in position and direction, for such an end, the deflection and slope are zero.

i.e., $y = 0$ and $\frac{dy}{dx} = 0$

(ii) Pinned end: In this case, the end is fixed in position only. i.e., $y = 0$

(iii) Free end: at such an end, the column is neither fixed in position nor in direction.

Types of columns

Based on the end conditions we have four important types of columns

1. Both end pinned
2. Both ends fixed
3. One end fixed and the other end pinned
4. One end fixed and the other end free (cantilever column)

1. COLUMNS WITH BOTH ENDS HINGED

Consider a column *AB* of length l hinged at both of its ends *A* and *B* *and* carrying a critical load P. As a result of loading, let the column deflect into a curved form AX_1B as shown in Fig. 10.8.

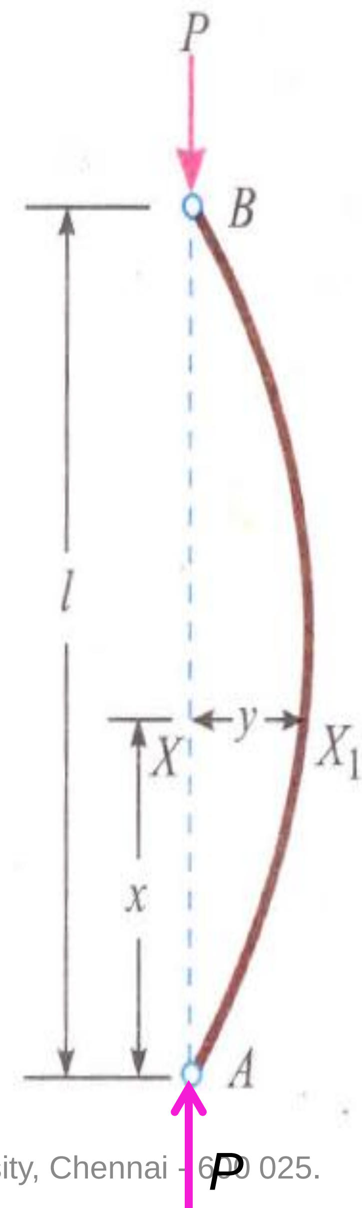


Fig. 10.8 (a)

Now consider a section X, at a distance *x* from the origin *A*.

Let, y = Deflection of the column at X.

$\therefore$Moment *at X*, *M* = $-Py$

$$\therefore EI\frac{d^2y}{dx^2} = -Py$$

i.e., $$\frac{d^2y}{dx^2} + \frac{Py}{EI} = 0$$

Let $\frac{P}{EI} = k^2$ $\qquad \therefore$ y''+k^2y = 0

i.e., $(D^2+k^2)y = 0$

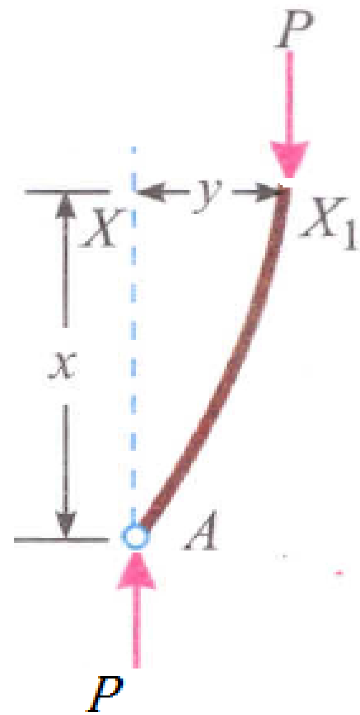


Fig. 10.8 (b)

Solution for the differential equation,

y = CF+PI

Complementary Function(CF)

Auxiliary equation is $m^2 + k^2 = 0$

$m = 0 \pm ki$

We know, If $(m = m_1 \pm m_2 i)$

$then \qquad CF = e^{m_1 x}(Acosm_2 x + Bsinm_2 \mathrm{x})$

$\therefore CF = Acos\,kx + Bsin\,kx$

Particular Integral(PI)

PI = 0

∴The general solution of the above differential equation is $y = A cos(kx) + B sin(kx)$

where A and B are the constants of integration that can be evaluated by applying the known boundary conditions.

We know that when $x = 0$; $y = 0 \Rightarrow \quad A = 0$

Similarly when $x = l$, then $y = 0 \Rightarrow$

$$0 = B\ sin(kl)$$

Now if we consider *B* to be equal to zero, then it indicates that the column has not bent at all. But if

$$sin(kl) = 0$$

$$kl = n\pi$$

$$kl = 0, \pi, 2\pi, 3\pi \ ...$$

Now taking the least significant value,

$$kl = \pi$$

$$k^2 l^2 = \pi^2$$

$$\therefore \qquad \boldsymbol{P = \frac{\pi^2 EI}{l^2}}$$

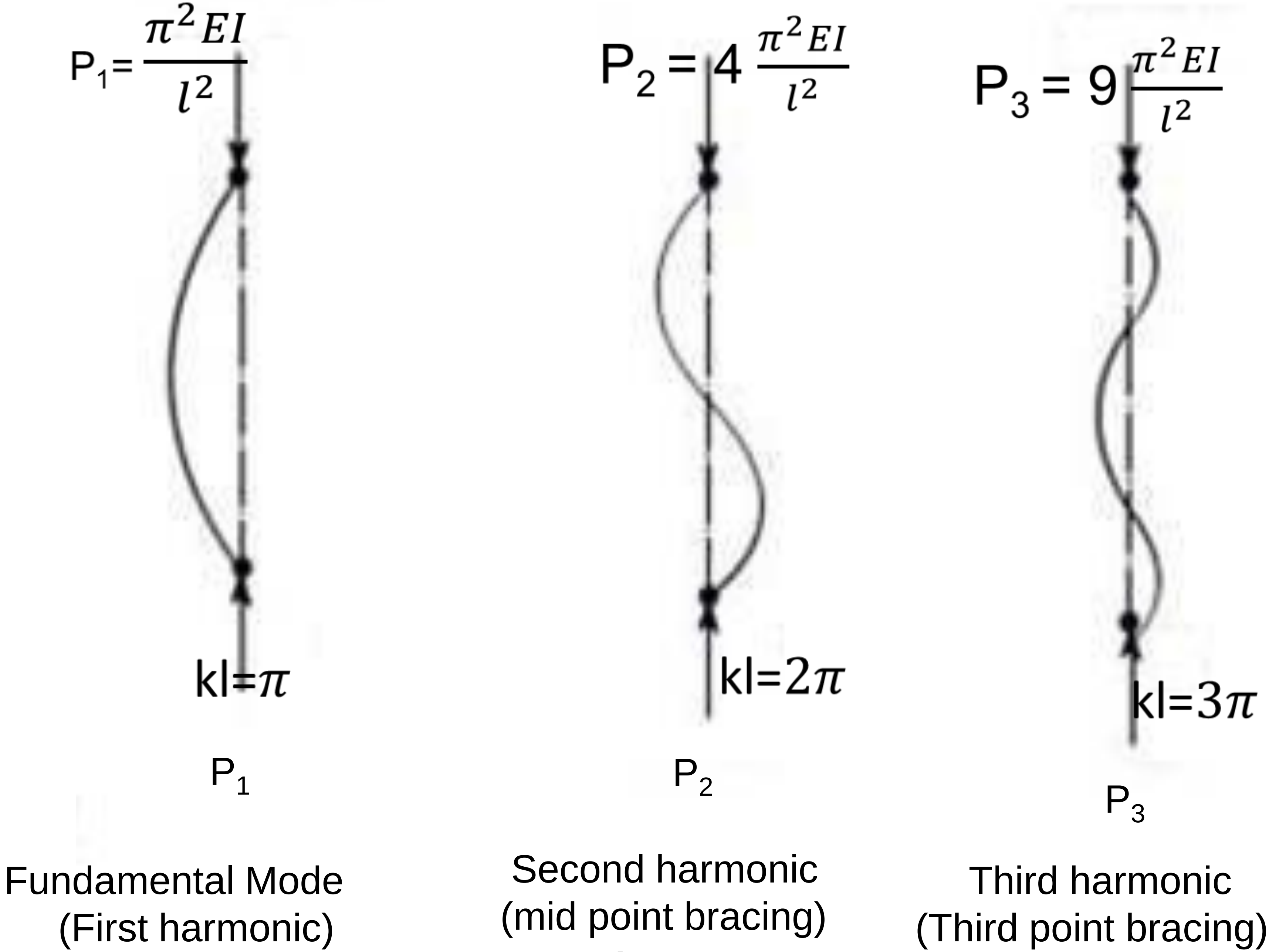

$P_1 = \frac{\pi^2 EI}{l^2}$
$P_2 = 4\frac{\pi^2 EI}{l^2}$
$P_3 = 9\frac{\pi^2 EI}{l^2}$
$kl=\pi$
$kl=2\pi$
$kl=3\pi$
P_1
P_2
P_3
Fundamental Mode
(First harmonic)
Second harmonic
(mid point bracing)
Third harmonic
(Third point bracing)

Fig. 10.9

2. COLUMNS WITH BOTH ENDS FIXED

Moments due to the critical load P at X,

$$M = M_0 - Py$$

$$EI\frac{d^2y}{dx^2} = M_0 - Py$$

$$\frac{d^2y}{dx^2} + \frac{Py}{EI} = \frac{M_0}{EI}$$

Let $\frac{P}{EI} = k^2$

$$\therefore \frac{d^2y}{dx^2} + k^2y = \frac{M_0}{P}k^2$$

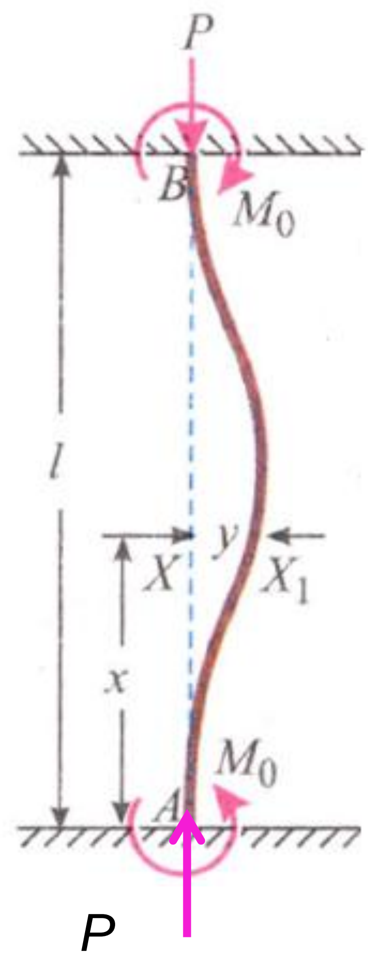


Fig. 10.10

Solution for the differential equation,

y = CF+PI

<u>Complementary Function(CF)</u>

Auxiliary equation is $m^2 + k^2 = 0$

$\therefore m = 0 \pm ki$

We know, If $(m = m_1 \pm m_2 i)$

$then \qquad CF = e^{m_1 x}(A cos m_2 x + B sin m_2 \mathrm{x})$

$\therefore CF = A cos\, kx + B sin\, kx$

and $\quad \text{Particular Integral}\,, PI = \dfrac{\frac{Mk^2}{P}}{D^2+k^2} = \dfrac{M}{P}$

∴The general solution of the above differential equation is:

$$y = A\cos(kx) + B\sin(kx) + \frac{M_0}{P} \text{ --------(i)}$$

where A and B are the constants of integration that can be evaluated by applying the known boundary conditions.

when x = 0, y = 0. $\Rightarrow A = -\frac{M_0}{P}$.

Now differentiating the equation (i),

$$\frac{dy}{dx} = -A\mathrm{k}\sin(kx) + Bk\cos(kx)$$

When x = 0, $\frac{dy}{dx} = 0. \Rightarrow 0 = Bk$

$$k \neq 0, \therefore B = 0$$

Substituting the values $A = -\frac{M_0}{P}$ and B=0 in equation (i)

$$y = -\frac{M_0}{P} cos(kx) + \frac{M_0}{P} = \frac{M_0}{P}[1 - cos(kx)]$$

We know, When x = l , then y = 0.

$$\therefore 0 = \frac{M_0}{P}[1 - cos(kl)]$$

$$\frac{M_0}{P} \neq 0 \qquad \therefore cos(kl) = 1$$

$\therefore \qquad \mathrm{k}l = 2n\pi$

i.e., $\mathrm{k}l = 0, 2\pi, 4\pi, 6\pi \ldots$

Now taking the least significant value,

$$kl = 2\pi$$

$$\boldsymbol{P = \frac{4\pi^2 EI}{l^2}}$$

(Or) $P = \frac{\boldsymbol{\pi^2 EI}}{(\boldsymbol{l}/\boldsymbol{2})^{\boldsymbol{2}}}$

3. COLUMNS WITH ONE END FIXED AND THE OTHER HINGED

M_A = Fixed end moment at A and

H = Horizontal reaction at A and *B as shown in Fig.10.11*

Moment at X due to critical load P,

$$M = -Py + H(l - x)$$

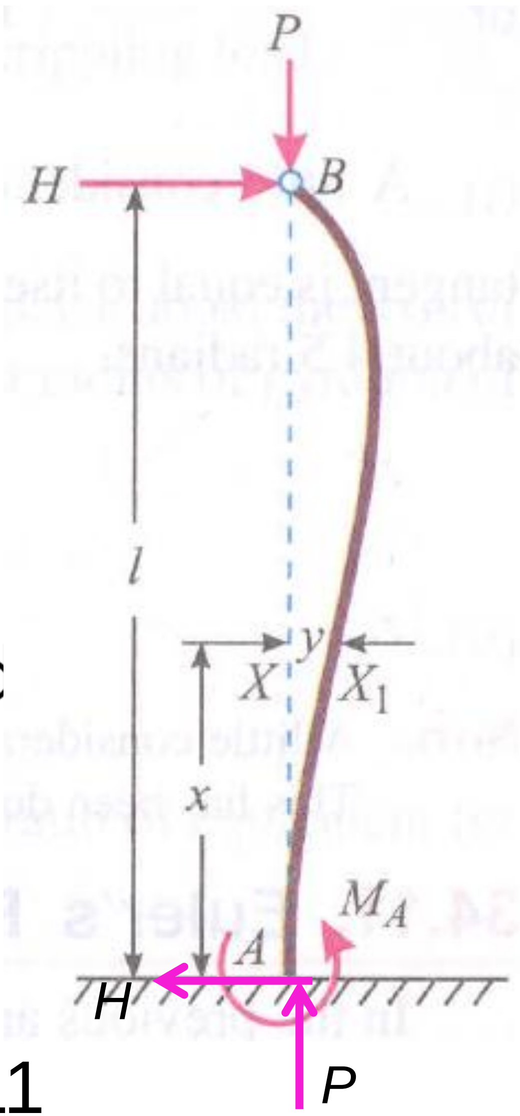


Fig. 10.11

$$\therefore EI\frac{d^2y}{dx^2} = H(l - x) - P.y$$

$$EI\frac{d^2y}{dx^2} + P.y = H(l - x)$$

Let $\frac{P}{EI} = k^2$

$$\therefore \frac{d^2y}{dx^2} + k^2y = \frac{Hk^2(l - x)}{P}$$

$$i.e., D^2 + k^2 = \frac{Hk^2(l-x)}{P}$$

Solution for the differential equation,

y = CF+PI

<u>Complementary Function(CF)</u>

Auxiliary equation is $m^2 + k^2 = 0$

$\therefore m = 0 \pm ki$

We know, If $(m = m_1 \pm m_2 i)$

$then \qquad CF = e^{m_1 x}(Acosm_2 x + Bsinm_2 \mathrm{x})$

$\therefore CF = Acos\,kx + Bsin\,kx$

Particular Integral, $PI = \dfrac{\frac{H}{P}(l-x)k^2}{D^2+k^2}$

$$= \frac{H}{P}(l - x)$$

∴The general solution is

$$y = A\,cos(kx) + B\,sin(kx) + \frac{H(l-x)}{P} \text{ ----(1)}$$

where *A* and *B* are the constants of integration that can be evaluated by applying the known boundary conditions.

When x = 0; y = 0 ⇒ A = - $\frac{Hl}{P}$

Differentiating equation (1) w.r.t x,

$$\frac{dy}{dx} = -A.k\,sin(kx) + Bkcos(kx) - \frac{H}{P} \text{ ------(2)}$$

When x = 0 , $\frac{dy}{dx} = 0. \Rightarrow 0 = Bk - \frac{H}{P}$

$$B = \frac{H}{Pk}$$

Substituting the value of A and B in (1) and put,

x = l , and y = 0

We get,

$$0 = -\frac{Hl}{P} cos(kl) + \frac{H}{Pk} sin(kl)$$

$$i.e., tan(kl) = (kl)$$

$$\Rightarrow \quad \mathrm{k}l = 4.5 \quad or \quad k^2 l^2 = 20.25 = 2\pi^2$$

$$\therefore \quad P = \frac{2\pi^2 EI}{l^2}$$

(or) $$P = \frac{\boldsymbol{\pi^2 EI}}{(\boldsymbol{l}/\sqrt{\mathbf{2}})^2}$$

4. COLUMNS WITH ONE END FIXED AND THE OTHER FREE

Moment at X,due to the critical load *P*,

$$M = P(\delta - y)$$

$$\therefore \qquad EI\frac{d^2y}{dx^2} = P\delta - P.y$$

$$or \qquad \frac{d^2y}{dx^2} + \frac{P}{EI}.y = \frac{P\delta}{EI}$$

$$let\ \frac{P}{EI} = k^2$$

$$\therefore (D^2+k^2)y = k^2\delta$$

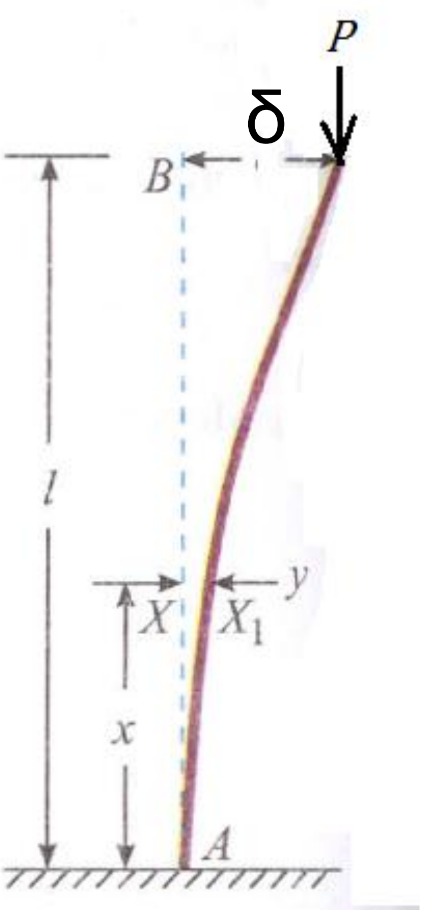


Fig. 10.12

Solution for the differential equation is,

y = CF+PI

Complementary Function(CF)

Auxiliary equation is $m^2 + k^2 = 0$

$\therefore m = 0 \pm ki$

We know, If $(m = m_1 \pm m_2 i)$

$then \qquad CF = e^{m_1 x}(Acosm_2 x + Bsinm_2 \mathrm{x})$

$\therefore CF = Acos\,kx + Bsin\,kx$

Particular Integral, $PI = \frac{ak^2}{D^2+k^2} = \delta$

∴The general solution is,

$y = A\,cos(kx) + B\,sin(kx) + \delta$ ----(1)

- where *A* and *B* are the constants of integration that can be evaluated by applying the known boundary conditions.
- We know that at $x = 0$, $y = 0$, $\Rightarrow$ $A = -\delta$

Differentiating equation (1) w.r.t x, gives

$$\frac{dy}{dx} = -Aksin(kx) + Bkcos(kx)$$

We know, when x = 0 , $\frac{dy}{dx} = 0 \Rightarrow 0 = kB$

$k \neq 0, \therefore$ B = 0

$$\therefore y = -\delta cos(kx) + \delta = \delta[1 - cos(kx)]$$

We know, When x = l , y = δ. Therefore

$$\delta = \delta[1 - cos(kl)]$$

$$1 = 1 - cos(kl)$$

$$cos(kl) = 0$$

$$\therefore \quad kl = \frac{n\pi}{2}$$

$$\mathrm{k}l = 0, \ \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2} \ldots$$

Now taking the least significant value,

$$kl = \frac{\pi}{2}$$

$$k^2 l^2 = \frac{\pi^2}{4}$$

$$\therefore \qquad P = \frac{\pi^2 EI}{4l^2}$$

(Or)

$$P = \frac{\pi^2 EI}{(2l)^2}$$

Equivalent length

We can write the general equation for Euler's critical load as

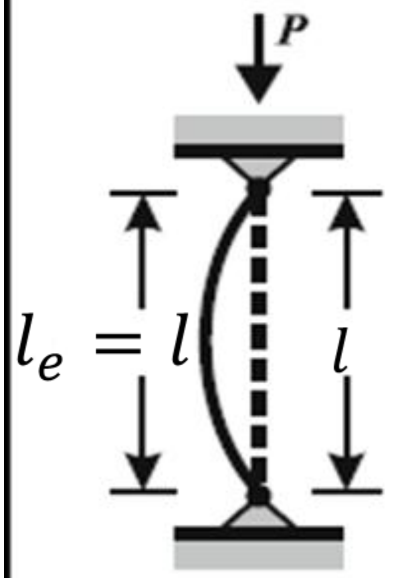
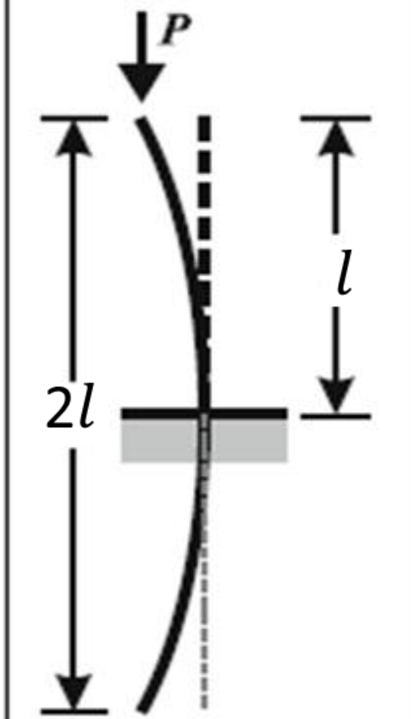
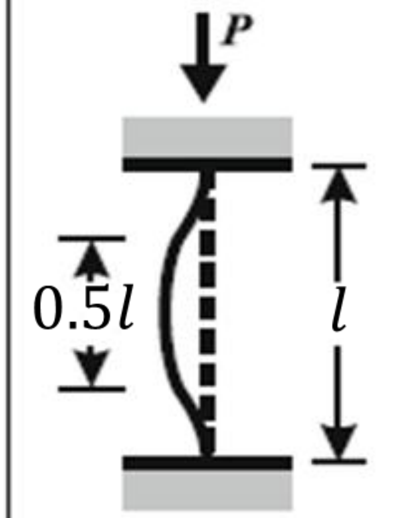
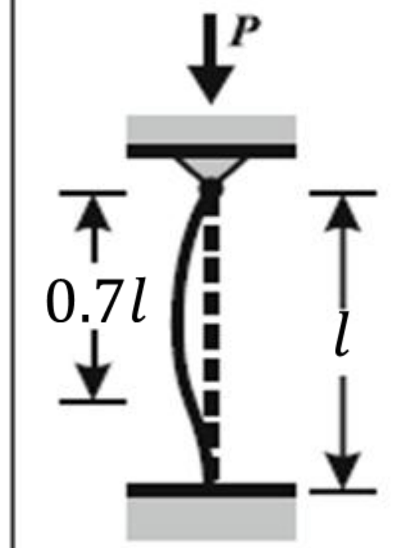
$$P = \frac{\pi^2 EI}{{l_e}^2}$$

Where, l_e is the equivalent length of a column, the distance between points on the column where the moment is zero, corresponding to the end conditions of the standard pinned–pinned column.

EQUIVALENT LENGTH (EFFECTIVE LENGTH)

- Definition:
- The equivalent length of a given column with given end conditions, is the length of an equivalent column of the same material and cross-section with both ends hinged and having the value of the crippling load equal to that of the given column.

Table 10.1. Effective Length for columns with common end conditions.

Pinned–pinned	Fixed–free	Fixed–fixed	Fixed–pinned
			

S.No	End Conditions	Relation between equivalent length(L_e) and actual length(l)	Crippling Load(P)
1.	Both ends hinged	$\boldsymbol{l_e = l}$	$P = \frac{\pi^2 EI}{(l)^2} = \frac{\pi^2 EI}{l^2}$
2.	One end fixed and other free	$\boldsymbol{l_e = 2l}$	$P = \frac{\pi^2 EI}{(2l)^2} = \frac{\pi^2 EI}{4l^2}$

S.No	End Conditions	Relation between equivalent length(L_e) and actual length(l)	Crippling Load(P)
3.	Both ends fixed	$\boldsymbol{l_e = \frac{l}{2}}$	$P = \frac{\pi^2 EI}{(\frac{l}{2})^2} = \frac{4\pi^2 EI}{l^2}$
4.	One end fixed and the other hinged	$\boldsymbol{l_e = \frac{l}{\sqrt{2}}}$	$P = \frac{\pi^2 EI}{(\frac{l}{\sqrt{2}})^2} = \frac{2\pi^2 EI}{l^2}$

- Physically, the *effective length* is the distance between points on the buckled column where the moment goes to zero, i.e., where the column is effectively pinned. Considering the *deflected shape*, the moment is zero where the curvature is zero (from beam theory).
- Zero curvature corresponds to an inflection point in the deflected shape (where the curvature changes sign).

Critical Column Stress

- A column can either fail due to the material yielding, or because the column buckles, it is of interest to the engineer to determine when this point of transition occurs.
- Consider the Euler buckling equation

$$P = \frac{\pi^2 EI}{(l_e)^2}$$

Critical Column Stress

- Because of the large deflection caused by buckling, the least moment of inertia *I* can be expressed as $I = Ar^2$
- where: *A* is the cross sectional area and *r* is the ***radius of gyration*** of the cross sectional area, i.e. . $r = \sqrt{\frac{I}{A}}$
- Note that the *smallest* radius of gyration of the column, i.e. the *least* moment of inertia *I* should be taken in order to find the critical stress.

Critical Column Stress

- Dividing the buckling load by A, gives:
- $f_c = \frac{P_E}{A} = \frac{\pi^2 E A r^2}{A {l_e}^2} = \frac{\pi^2 E}{(l_e/r)^2}$
- where:

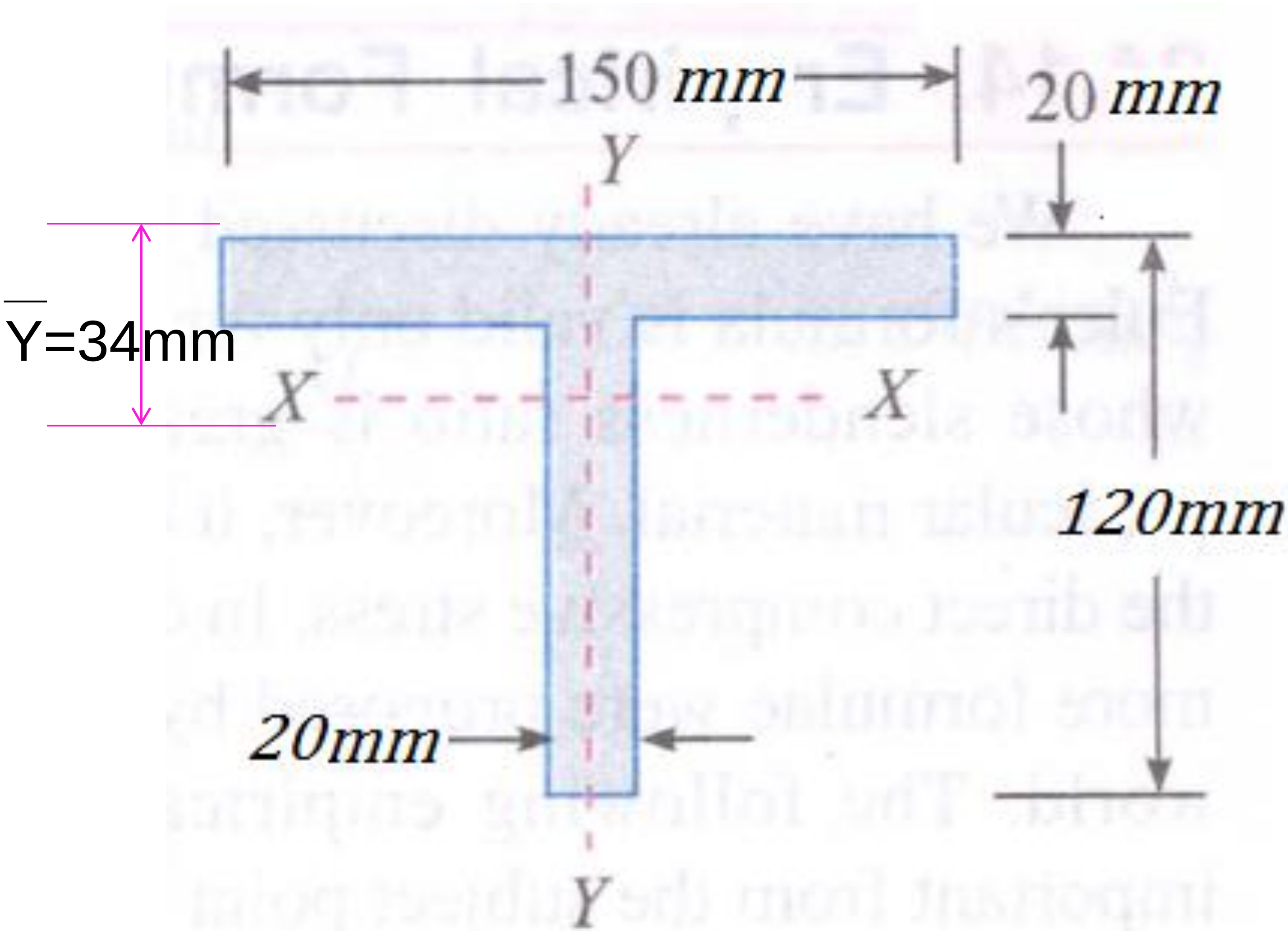
f_c is the compressive stress in the column and must not exceed the yield stress f_y of the material, i.e. $f_c < f_Y$,

l_e/r is called the ***slenderness ratio***, it is a measure of the column's flexibility.

Limitations for the use of Euler's formula

1. It is applicable to an ideal strut only and in practice, there is always crookedness in the column and the load applied may not be exactly co-axial.
2. It takes no account of direct stress. It means that it may give a buckling load for struts, far in excess of load which they can be withstand under direct compression.

Ex.10.1 A T section $150\,mm \times 120\,mm \times 20\,mm$ is used as a strut of 4m long with hinged at its both ends. Calculate the crippling load if modulus of elasticity for the material be $2.0 \times 10^5 N/mm^2$


150 mm
20 mm
Y
$\bar{Y}$=34mm
X
X
120mm
20mm
Y

Solution

First of all, let us find out the C.G of the section. Let $\bar{y}$ be the distance between C.G of the section from top of the flange

By geometry of the figure, $\bar{x} = 0$

$$\bar{y} = \frac{a_1 y_1 + a_2 y_2}{a_1 + a_2}$$

$$= \frac{(150 \times 20) \times 10 + (100 \times 20) \times 70}{(150 \times 20) + (100 \times 20)} = 34\, mm$$

$$I_{XX} = \left[\frac{1}{12} \times 150 \times 20^3 + (150 \times 20) \times 24^2\right] + \left[\frac{20 \times 100^3}{12} + 2000 \times 36^2\right]$$

$$= 6.0867 \times 10^6 mm^4$$

And

$$I_{YY} = \frac{20 \times 150^3}{12} + \frac{100 \times 20^3}{12}$$

$$= 5.6917 \times 10^6 mm^4$$

Since I_{YY} is less than I_{XX}.

$\therefore$ The column will tend to buckle in y-y direction

Given End condition : both ends hinged

$$\therefore \quad l_e = l$$

$$\therefore \quad P = \frac{\pi^2 EI}{l^2} = \frac{\pi^2 \times 2.0 \times 10^5 \times 5.6917 \times 10^6}{4000^2}$$

$$= 702185\ N$$

10.2 Compare the ratio of the strength of a solid steel column to that of a hollow of the same cross-sectional area. The internal diameter of the hollow column is *3/4 of the external diameter. Both the columns have the same length and are pinned at both ends.*

Solution:

Euler's crippling load for the solid column,

$$P_S = \frac{\pi^2 EI}{{l_s}^2} = \frac{\pi^2 E A_s {K_s}^2}{l^2}$$

Euler's crippling load for the hollow column:

$$P_E = \frac{\pi^2 EI}{{l_H}^2} = \frac{\pi^2 E A_H {K_H}^2}{l^2}$$

$$\frac{P_H}{P_S} = \left(\frac{k_H}{k_S}\right)^2 = \frac{\dfrac{\dfrac{\pi(D^4 - d^4)}{64}}{\dfrac{\pi(D^2 - d^2)}{4}}}{\dfrac{\dfrac{\pi({D_1}^4)}{64}}{\dfrac{\pi({D_1}^2)}{4}}}$$

$$\frac{\dfrac{D^2 + d^2}{16}}{\dfrac{{D_1}^2}{16}} = \frac{D^2 + d^2}{{D_1}^2} = \frac{D^2 + (\frac{3D}{4})^2}{{D_1}^2} = \frac{25D^2}{16{D_1}^2}$$

Since the cross-sectional areas of the both the columns are equal, therefore

$$\frac{\pi}{4} \times {D_1}^2 = \frac{\pi}{4}(D^2 - d^2) = \frac{\pi}{4}\left[D^2 - \left(\frac{3D}{4}\right)^2\right]$$

$$= \frac{\pi}{4} \times \frac{7D^2}{16}$$

$$\therefore \qquad {D_1}^2 = \frac{7D^2}{16}$$

$$\frac{P_H}{P_S} = \frac{25D^2}{16 \times \frac{7D^2}{16}} = \frac{25}{7} \qquad (Ans)$$

Rankine's Formula for columns

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E}$$

P_R = Crippling load by Rankine's Formula

$P_C = f_C . A$

= Ultimate crushing load for the column

$$P_E = \frac{\pi^2 EI}{{l_e}^2}$$

= Crippling load by Euler's formula

$$\frac{1}{P_R} = \frac{1}{P_C} + \frac{1}{P_E} = \frac{P_E + P_C}{P_C . P_E}$$

$$P_R = \frac{P_C . P_E}{P_C + P_E} = \frac{P_C}{1 + \frac{P_C}{P_E}}$$

$$P_R = \frac{f_C . A}{1 + f_C . A \times \frac{{l_e}^2}{\pi^2 E}} = \frac{f_C . A}{1 + \frac{f_C}{\pi^2 E} \times \frac{A{l_e}^2}{Ar^2}}$$

$$P_R = \frac{f_C . A}{1 + a\left(\frac{l_e}{r}\right)^2} = \frac{P_C}{1 + a\left(\frac{l_e}{r}\right)^2}$$

$P_C = Crushing\ load\ of\ the\ column\ material$

$f_C = Crushing\ stress\ of\ the\ column\ material$

$A = Cross - sectional\ area\ of\ the\ column$

$a = Rankine's\ constant\ \left(= \frac{f_C}{E\ \pi^2}\right)$

$L_e = Equivalent\ length\ of\ the\ column,$

r= $Least\ radius\ of\ gyration$

S.No	Material	$f_c in\ MPa$	$a = \frac{f_c}{\pi^2 E}$
1	Mild steel	320	$\frac{1}{7500}$
2	Cast iron	550	$\frac{1}{1600}$
3	Wrought iron	250	$\frac{1}{9000}$
4	timber	40	$\frac{1}{750}$

Ex. 10.3 A round steel rod of diameter 15 mm and length 2 m is subjected to a gradually increasing axial compressive load. Using Euler's formula find the buckling load. Find also the maximum lateral deflection corresponding to the buckling condition. Both ends of the rod may be taken as hinged. Take $E = 2.1 \times 10^5\ N/mm^2$ and the yield stress of steel = $240\ N/mm^2$

Solution:

Area of the rod $A = \frac{\pi d^2}{4} = \frac{\pi \times 15^2}{4} = 176.7\ mm^2$

Moment of Inertia of the section,

$$I = \frac{\pi d^4}{64} = \frac{\pi \times 15^4}{4} = 2485\ mm^4$$

Since both ends of the member are hinged,

The effective length, $l_e = 2000\ mm$

$\therefore$ Buckling load, $P = \frac{\pi^2 EI}{L^2} = \frac{\pi^2 \times 2.1 \times 10^5 \times 2485}{2000^2}$

=1287.61N

Direct compressive stress,

$$f_C = \frac{P}{A} = \frac{1287.61}{176.7} = 7.287\ N/mm^2$$

Let the maximum bending stress corresponding to buckling condition be f_b

$$\therefore \qquad f_b + f_C = \text{yeild stress}$$

$$f_b = 240 - 7.287 = 232.713\ N/mm^2$$

Let M be the maximum bending moment which occurs as the Centre

$$\therefore \quad \frac{M}{I} = \frac{f_b}{\left(\frac{d}{2}\right)} \qquad \Bigg| \qquad \frac{M}{I} = \frac{f}{y}$$

$$M = \frac{232.713 \times 2485}{7.5} = 77106\ N\ mm$$

Let the maximum central deflection be δ mm

$$M = P \times \delta = 1287.61\delta$$

$$\delta = \frac{77106}{1287.61} = 59.9\ mm \qquad (Ans)$$

Ex.10.4 A cast iron hollow cylindrical column 3m in length when hinged at both the ends, has a critical buckling load of P KN. When this column is fixed at both the ends , its critical load rises to P + 300 kN .If the ratio of external diameter to internal diameter is 1.25 and E=1.0$\times 10^5\ N/mm^2$. Determine the external diameter of the column.

$$d_0 = outer\ diameter\ of\ column$$

$$d_i\ = inner\ diameter\ of\ column$$

$$\frac{d_0}{d_i} = 1.25, \qquad\qquad l = 3000\ mm$$

$$I = \frac{\pi}{64}\left({d_0}^4 - {d_i}^4\right) = \frac{\pi}{64}\left((1.25d_i)^4 - {d_i}^4\right)$$

$$= \frac{1.44\pi {d_i}^4}{64}$$

Both ends are fixed. $l_e = l$

$$P = \frac{\pi^2 EI}{{l_e}^2} = \frac{\pi^2 \times 1.0 \times 10^5 \times 1.44\pi \times {d_i}^4}{64 \times 3000^2}$$

$$= 0.00025\pi^3 {d_i}^4 \_\_\_\_\_\_\_\_\_\_ (1)$$

When both ends are fixed. $L_e = l/2 = 1500\ cm$

$$P + 300 \times 10^3 = \frac{\pi^2 EI}{{l_e}^2}$$

$$= \frac{\pi^2 \times 1.0 \times 10^5 \times 1.44\,\pi \times {d_i}^4}{64 \times 1500^2}$$

$$= 0.001\pi^3 {d_i}^4$$

$$\therefore P = 0.001\pi^3 {d_i}^4 - 300 \times 10^3 \_\_\_\_\_\_\_\_(2)$$

Equating (1) & (2)

$$0.00025\pi^3 {d_i}^4 = 0.001\pi^3 {d_i}^4 - 300 \times 10^3$$

$$0.00075\pi^3 {d_i}^4 = 300 \times 10^3$$

$$d_i = 59.93\ mm$$

$\therefore$ External diameter is = 1.25 $\times$ 59.93 = 74.91 mm

Ex.10.5 A hollow cylindrical cast iron column is 4m long, both ends being fixed. Design the column to carry an axial load of 250 kN. Use Rankin's formula and adopt a factor of safety of 5 . The internal diameter may be taken as 0.80 times the external diameter.

Take $f_c = 550\ N/mm^2$ and $a = \frac{1}{1600}$

Solution:

Let the external diameter be D mm

Internal diameter d = 0.80 D mm

Area of cross section = $\frac{\pi}{4}(D^2 - d^2)$

$$= \frac{\pi}{4}(D^2 - (0.8D)^2) = 0.2827D^2$$

MI of the Section $I = \frac{\pi}{64}(D^4 - d^4)$

$$= \frac{\pi}{4}(D^4 - (0.8D)^4) = 0.029D^4$$

$$\therefore \qquad r = \sqrt{\frac{I}{A}} = \sqrt{\frac{0.029D^4}{0.2827D^2}} = 0.32D$$

Safe load on the column= 250 000N

∴ Crippling load = Safe load × factor of safety

= 250 000 × 5 = 1.25× $10^6 N$

Both end being fixed,

Effective length of the Column

$$l_e = \frac{l}{2} = \frac{4000}{2} = 2000\ mm$$

Applying Rankine's formula

$$P = \frac{f_c A}{1 + a(\frac{l_e}{r})^2}$$

i.e., $$1250000 = \frac{550 \times 0.2827 D^2}{1 + \frac{1}{1600}\left(\frac{2000}{0.32D}\right)^2}$$

$$1250000 = \frac{155.485 D^2}{1 + \frac{24414}{D^2}}$$

Solving, $D = 136.3\ mm \quad (say\ 140\ mm)$

$$Internal\ Diameter = 0.8 \times 136.3\ mm$$

$$= 109.07\ cm \ \ (say\ 110\ mm)$$

Ex.10.6 A column of 9 m long has a cross section shown in figure. The column is pinned at both ends. If the column is subjected to an axial load equal in value ¼ of the Euler's critical load for the column. Determine the factor of safety on the Rankine's ultimate stress value.

Take $f_c = 326\ N/mm^2$,

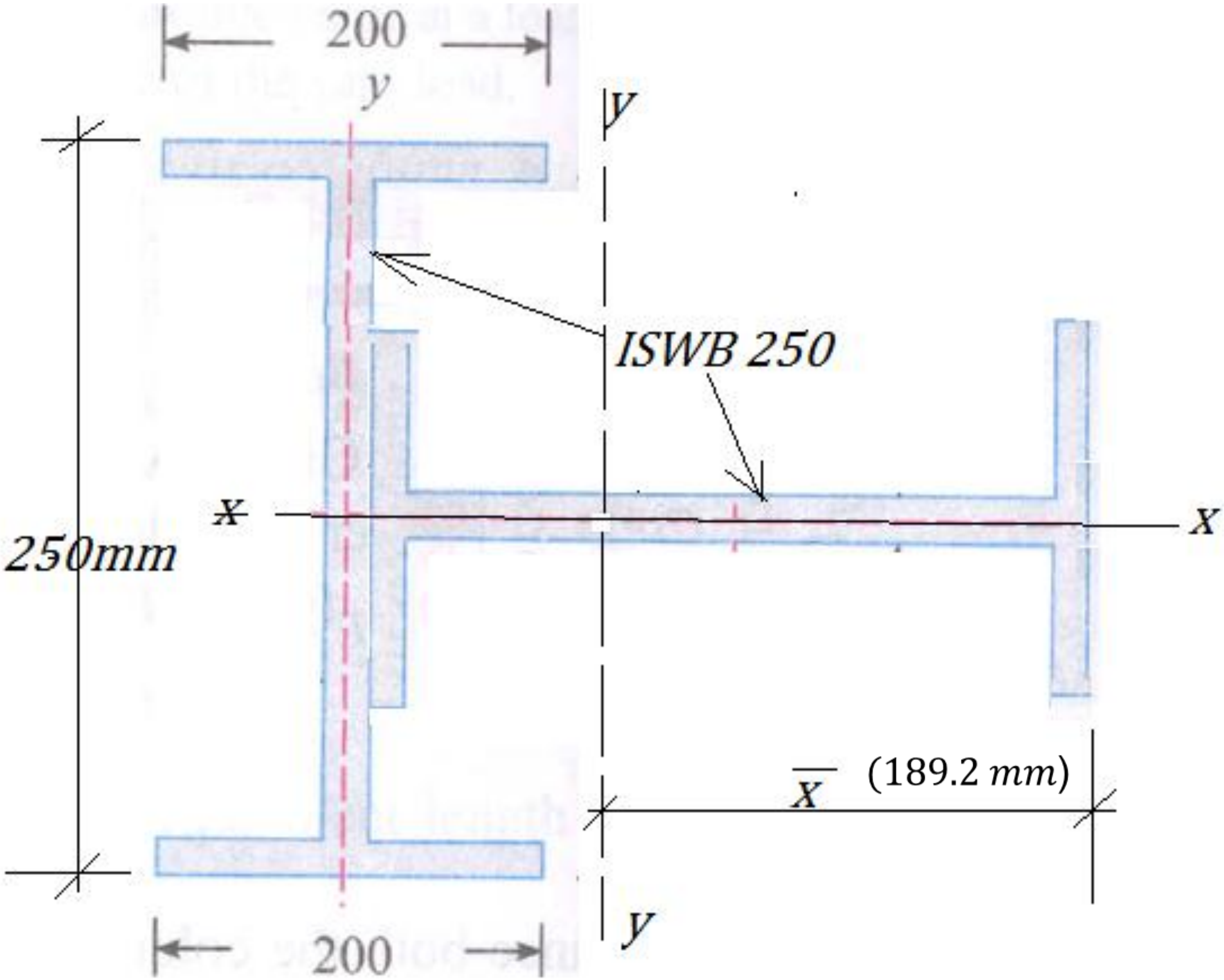
Rankin's constant $a = \frac{1}{7500}$, $E = 200$ GPa

properties of one RSJ Area = 5205 mm^2

$$I_{XX} = 5943.1 \times 10^4\ cm^4$$

$$I_{YY} = 857.5 \times 10^4 cm^4$$

Thickness of the web = 6.7 mm


200
y
y
ISWB 250
x
x
250mm
$\overline{X}$ (189.2 mm)
200
y

Solution:

First of all find the C.G of the combined section. From the geometry of the fig. $\bar{y} = 0$

$$\bar{x} = \frac{a_1 x_1 + a_2 x_2}{a_1 + a_2}$$

$$= \frac{5205 \times 253.35 + 5205 \times 125}{10410}$$

$$= 189.2\, mm$$

Area of Combined section A= 10410 mm^2

MI about xx axis,

$$I_{XX} = 5943.1 \times 10^4 + 857.5 \times 10^4$$
$$= 6800.6 \times 10^4 mm^4$$

$$I_{YY} = 857.5 \times 10^4 + 5205 \times 64.15^2 + 5943.1 \times 10^4 + 5205 \times 64.2^2$$

$$= 11087.89 \times 10^4 mm^4$$

$I_{XX} < I_{YY}$

$$r^2 = \frac{I}{A} = \frac{6800.6 \times 10^4}{10410} = 6533\ mm^2$$

Euler's critical Load $= \dfrac{\pi^2 EI}{{l_e}^2}$

l_e (for both ends pinned) = l = 9000mm

$$\therefore \quad P_E = \frac{\pi^2 \times 200 \times 10^3 \times 6800.6 \times 10^4}{9000^2}$$

$$= 1657.3 \times 10^3 \mathrm{N}$$

Safe load based on Euler′s theory,

$$= \frac{1657.3}{4} = 414.3\ kN$$

Crippling load as per Rankin's formula

$$P = \frac{f_c A}{1 + a(\frac{l_e}{r})^2} = \frac{326 \times 10410}{1 + \frac{1}{7500} \times \frac{9000^2}{6533}}$$

$$= 1279.1 \times 10^3 \text{ N}$$

Safe load = $414.3\ kN$

∴ Factor of safety on Rankin's ultimate stress value

$$= \frac{1279.1}{414.3} = 3.09$$

Ex. 10.7 A compound stanchion is made up of two ISMC 250 placed back to back with a gap between adjacent flat surfaces. Two 360mm × 12 mm plates are riveted to the flanges, so as to form a symmetrical box section. Determine the amount of gap if the column is to carry the maximum load.

Properties of one ISMC 250 are:

Area = 3867 mm^2

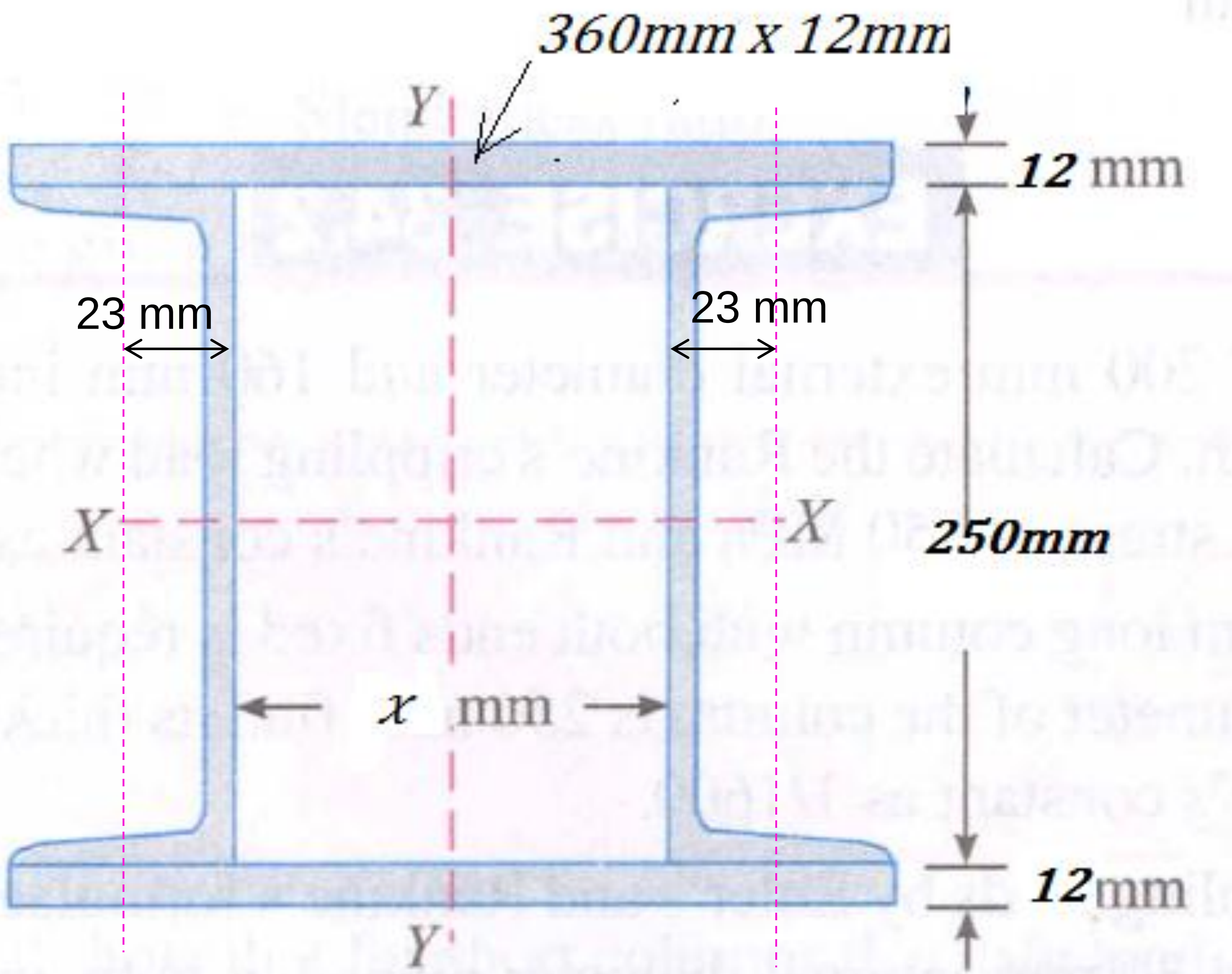
$Max\ MI(I_{xx1}) = 3816.1 \times 10^4 mm^4$

$Min\ MI(I_{yy1}) = 219.1 \times 10^4 mm^4$

$\text{Distance of the centroid from the back} = 23\ \text{mm}$

If the effective length of the stanchion is 8.5 m, calculate the safe maximum load, the working stress being interpolated from the following table

$\frac{l_e}{r}$	**20**	**40**	**60**	**80**	**100**	**120**	**140**
Working stress I N/mm^2	123.9	120.3	113.0	100.7	84.0	67.1	53.1



Fig

The stanchion shall carry maximum load when $I_{XX} = I_{YY}$.

For the given section

$$I_{XX} = 2 \times 3816.8 \times 10^4 + 2\left(\frac{360 \times 12^3}{12} + 360 \times 12 \times 131^2\right)$$

$$= 22471.1 \times 10^4 cm^4$$

If the amount of gap be $'x'$ then

$$I_{Yy} = 2\left[\frac{12 \times 360^3}{12} + 219.1 \times 10^4 + 3867(\frac{x}{2} + 23)^2\right]$$

$$= 9769.4 \times 10^4 + 19335(x + 46)\ ^2$$

But $\quad I_{XX} = I_{YY} \Rightarrow$

$$9769.4 \times 10^4 + 19335(x + 46)^2 = 22471.1 \times 10^4$$

Solving , we get

$$x = 210.3\ mm$$

Cross section area of stanchion

$$= (2 \times 3867) + (2 \times 360 \times 12)$$

$$= 16374\ mm^2$$

$$r = \sqrt{\frac{I}{A}} = \sqrt{\frac{22471.1 \times 10^4}{16374}} = 117.1\ mm$$

$$\therefore slenderness\ ratio, \frac{l_e}{r} = \frac{8500}{117.1} = 72.56$$

From the given table, the working stress corresponding to the calculated slenderness ratio by interpolation is

$$p_c = 113 - \frac{113 - 100.7}{80 - 60}(72.56 - 60)$$

$$= 105.276\, N/mm^2$$

Safe maximum load $P = p_c \times A$

$$= 105.276 \times 16374$$

$$= 1723780N$$

$$= 1723.78\, kN$$

UNIT – V

STRESSES IN TWO DIMENSIONS

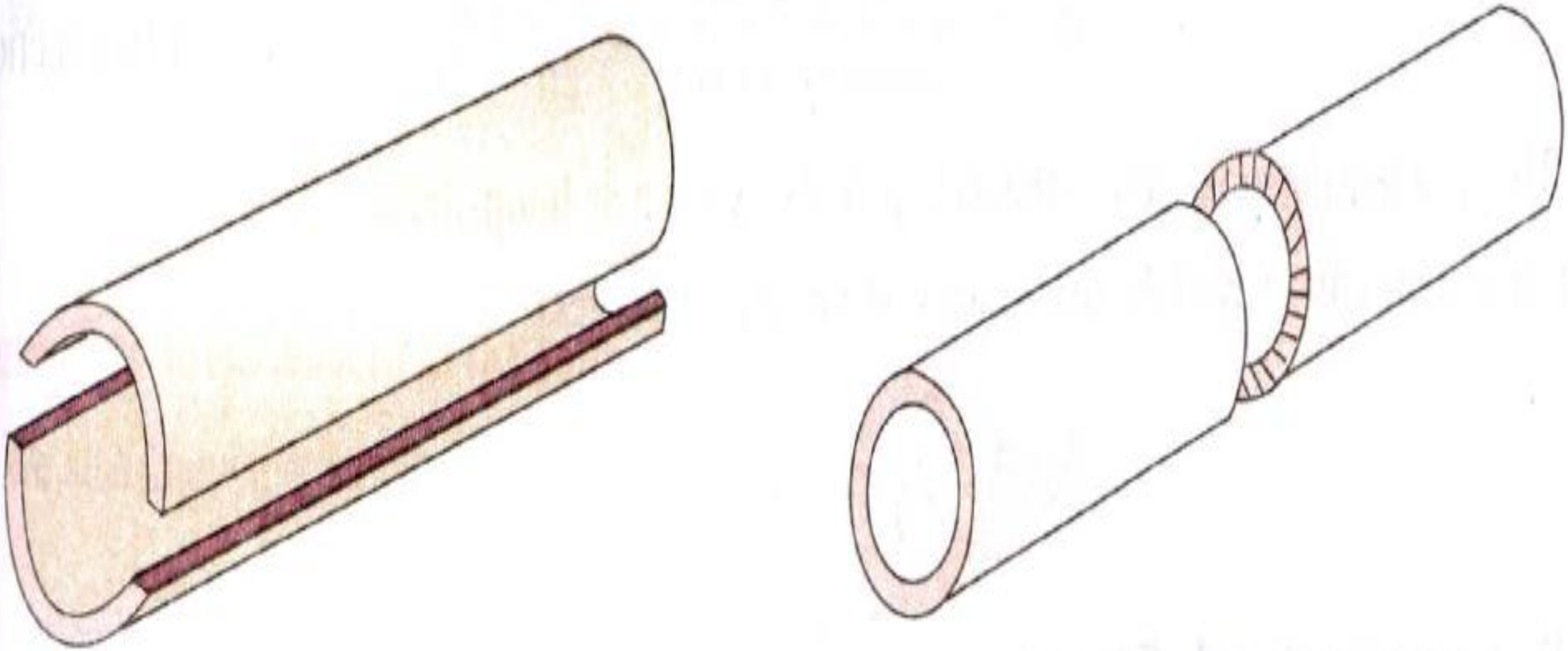
THIN SHELLS

- In engineering field, we daily come across vessels of cylindrical and spherical shapes containing fluids such as pipes, tanks, boilers, compressed air receivers etc.
- Generally, the walls of such vessels are very thin as compared to their diameters.

- In general, if the thickness of the wall of a shell is less than 1/l0th to 1/15th (approximately 7%) of its diameter, it is known as a thin shell.

- These vessels, when empty, are subjected to atmospheric pressure internally as well as externally. In such a case, the resultant pressure on the walls of the shell is zero.

- whenever a cylindrical shell is subjected to an internal pressure, its walls are subjected to tensile stresses.
- It will be interesting to know that if these stresses exceed the permissible limit, the cylinder is likely to fail in anyone of the following two ways as shown in Fig. 11.1 *(a)* and *(b).*



(*a*) Split into two troughs. (*b*) Split into two cylinders.

Fig.11.1

Stresses in a Thin Cylindrical Shell

The walls of the cylindrical shell will be subjected to the following two types of tensile stresses:

1. Circumferential stress and
2. Longitudinal stress.

In case of thin shells, the stresses are assumed to be uniformly distributed across the wall thickness.

However, in case of thick shells, the stresses are no longer uniformly distributed across the thickness and the problem becomes complex.

Circumferential Stress

- Consider a thin cylindrical shell subjected to an internal pressure as shown in Fig.11.2(b)
- We know that as a result of the internal pressure, the cylinder has a tendency to split up into two troughs Fig.11.2(a)

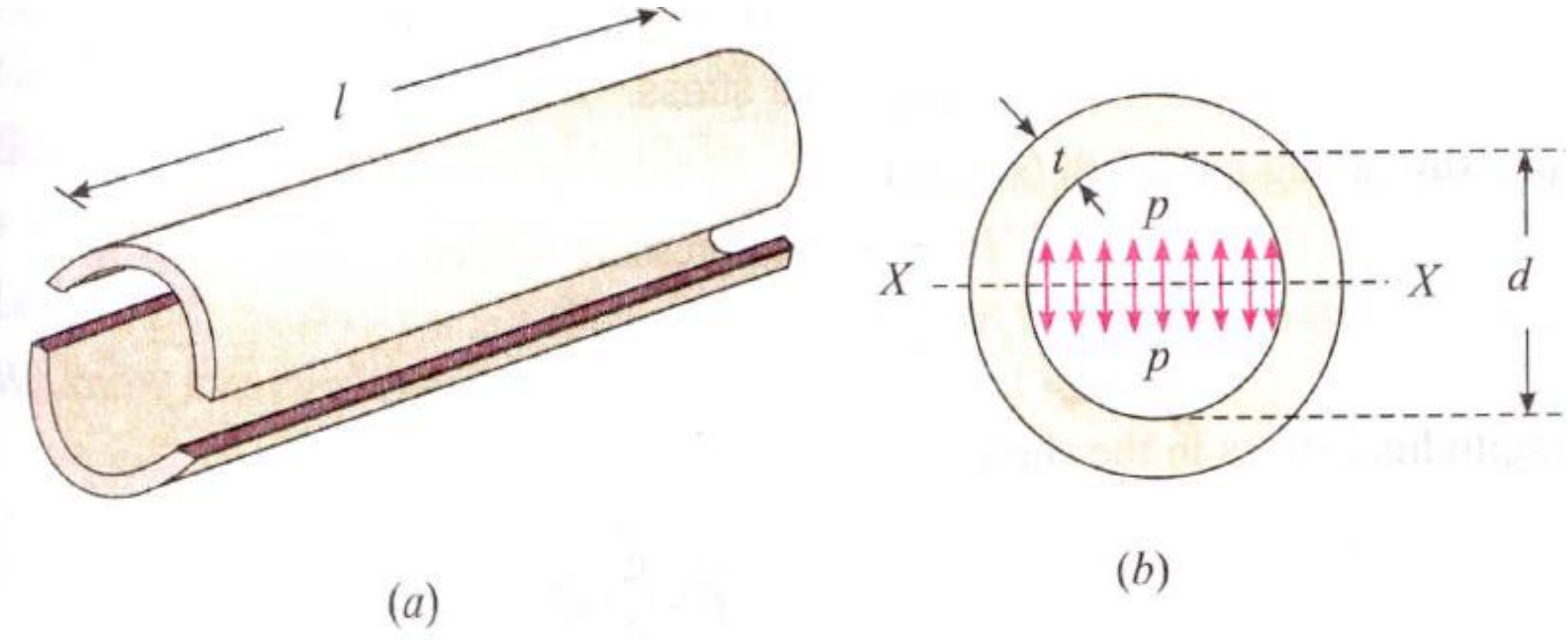


Fig.11.2. Failure of thin cylinder

The force tending to push the two halves apart is given by :

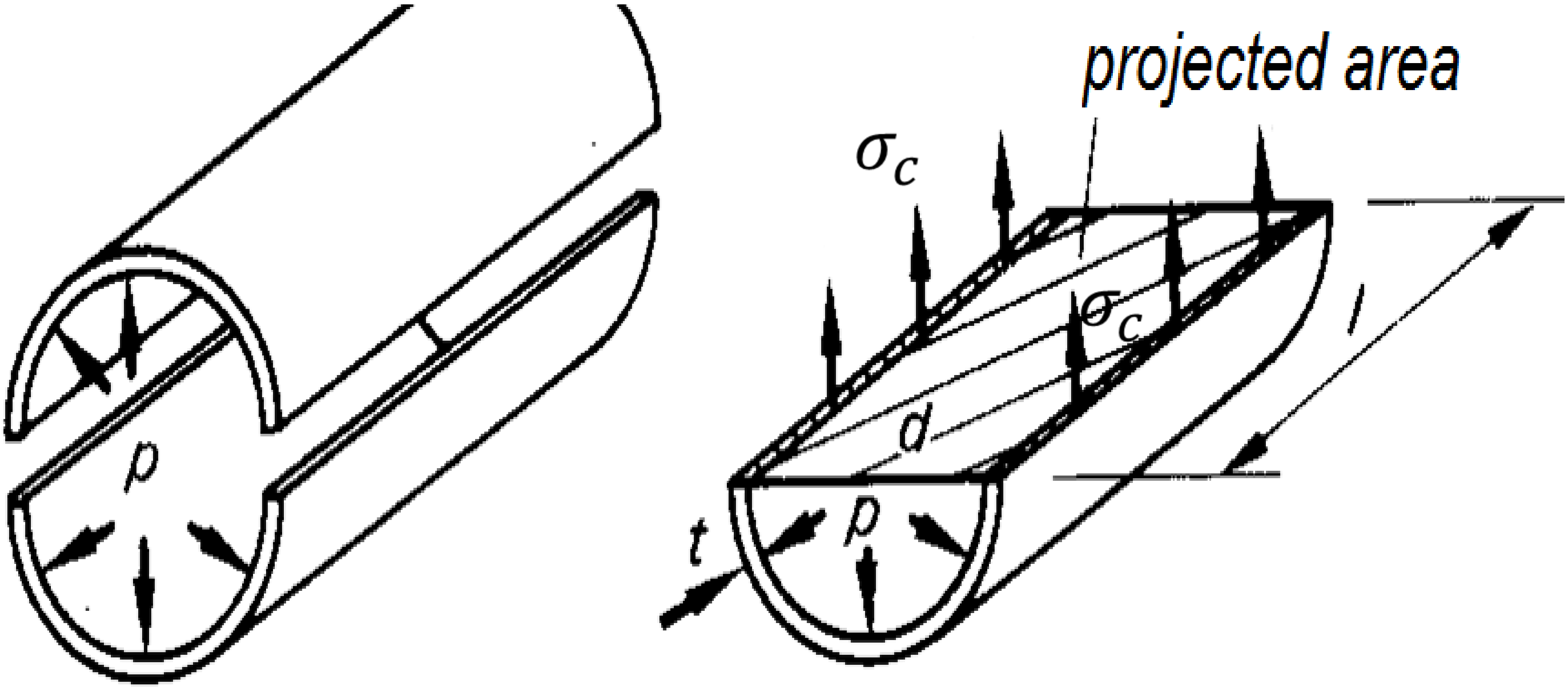
$$F = internal\ pressure\ \times projected\ area$$

$$= p\ \times (d \times l)$$

The cross-section area of material which sustains this force is

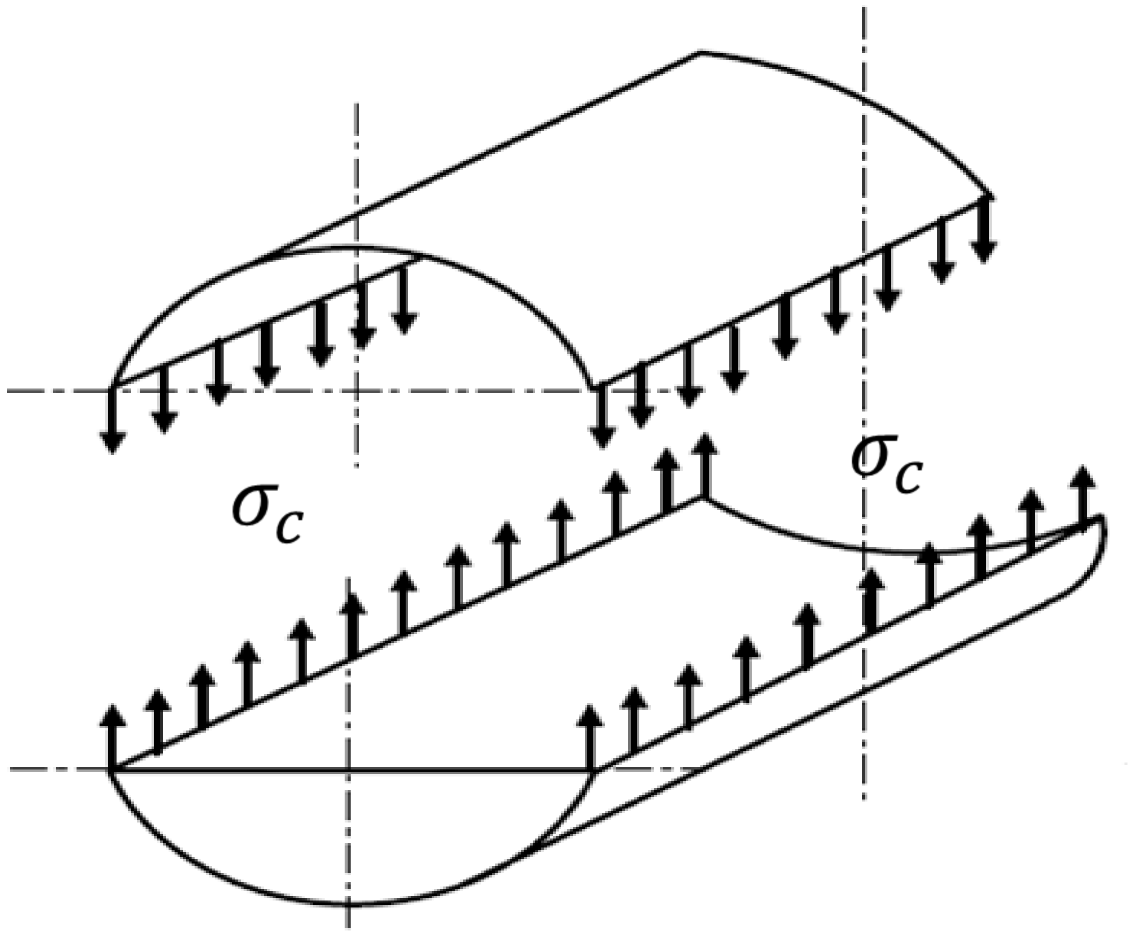
given by:

$$A = 2tl$$



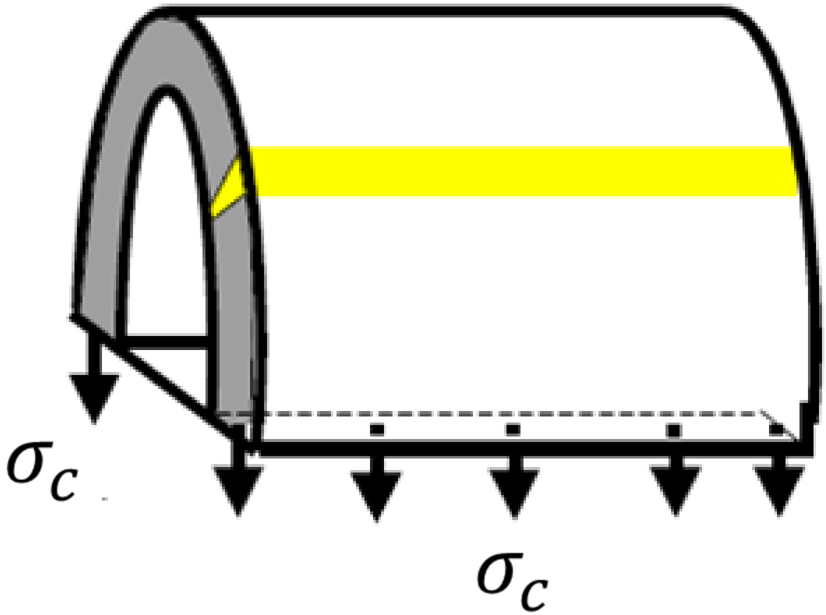
Failure along longitudinal section

Fig.11.3

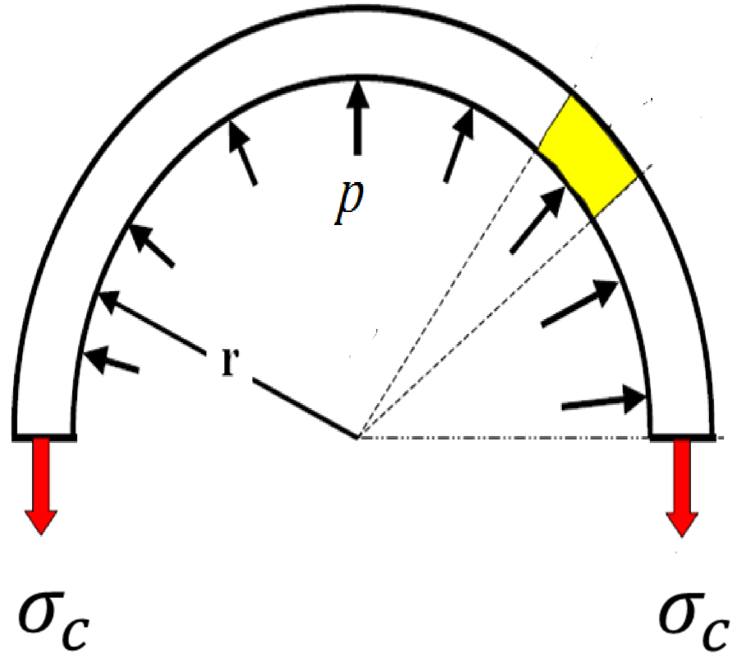


(a)

Fig.11.4. Circumferential stress



(b)



(c)

Fig.11.4. Circumferential stress

Therefore the circumferential stress is given by:

$$\sigma_c = \frac{Total\, pressure}{Resisting\ section} = \frac{pdl}{2tl} = \frac{pd}{2t}$$

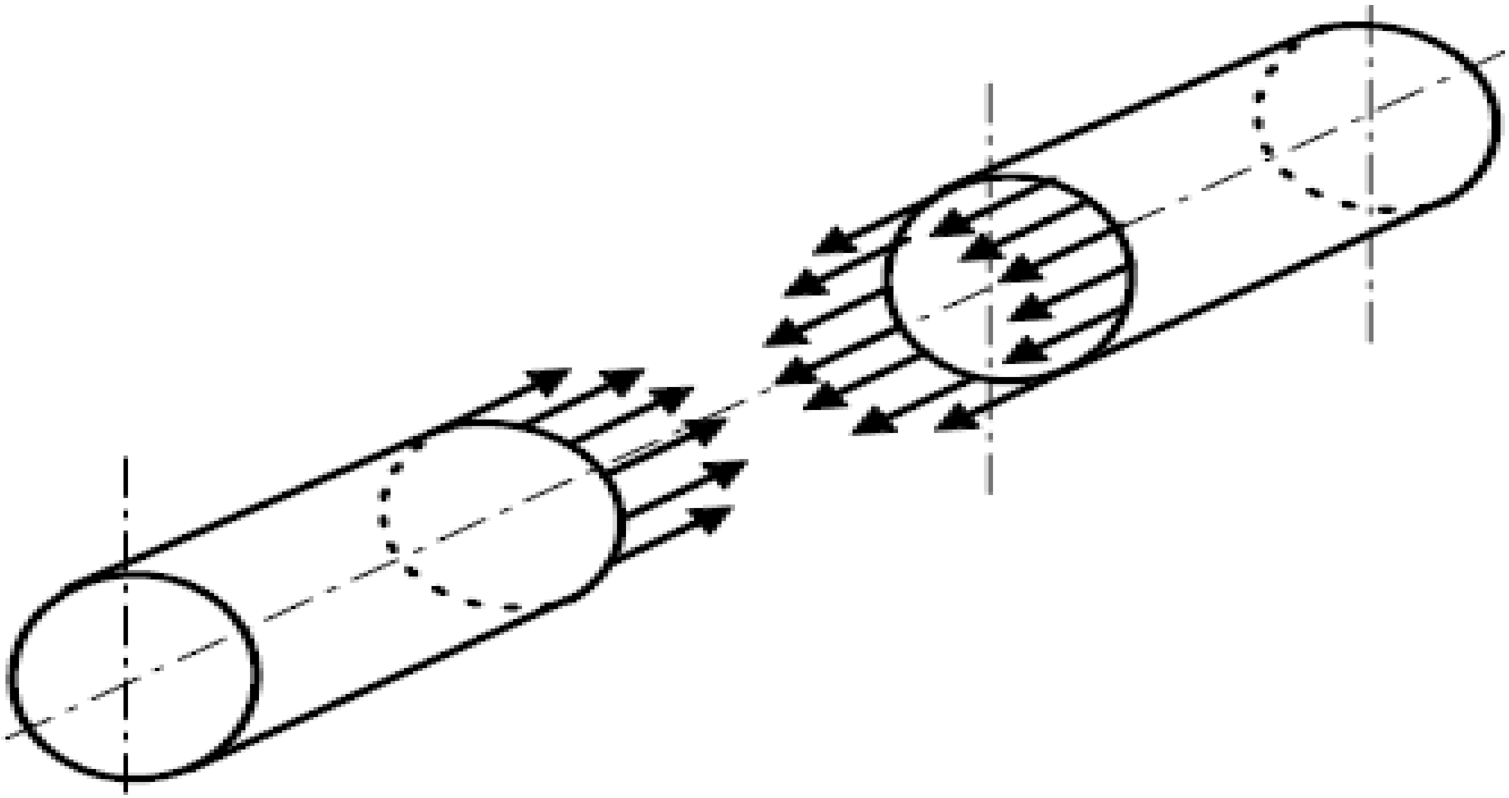
Note:

If η is the efficiency of the riveted joints of the shell, then stress,

$$\sigma_c = \frac{pd}{2t\eta}$$

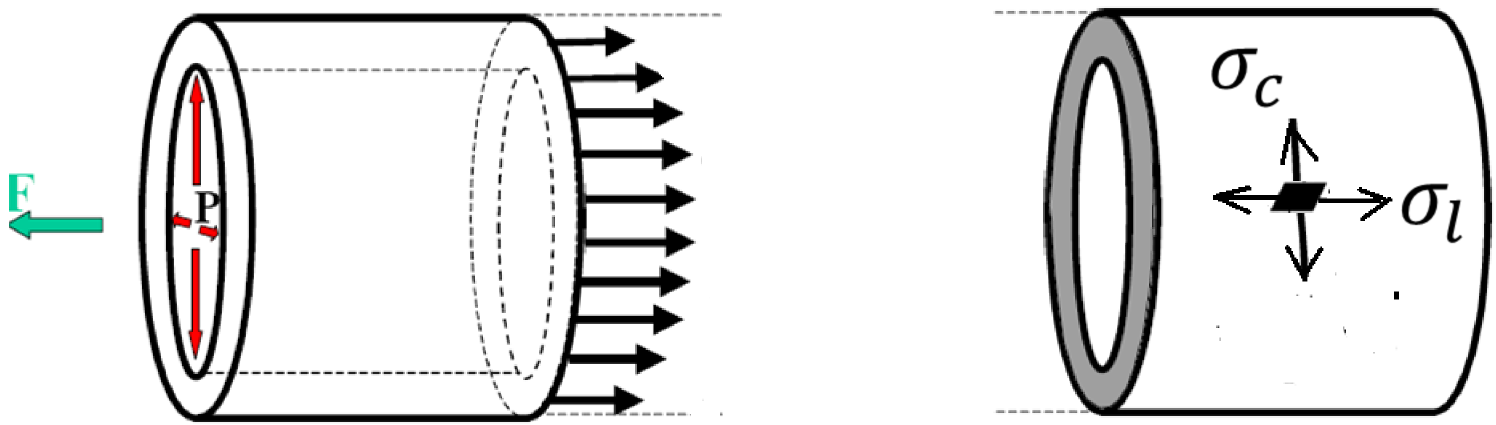
Longitudinal (or Axial) stress

- Consider the same cylindrical shell, subjected to the same internal pressure as shown in Fig.11.5
- We know that as a result of the internal pressure, the cylinder also has a tendency to split into two pieces as shown in the figure.11.5(a)

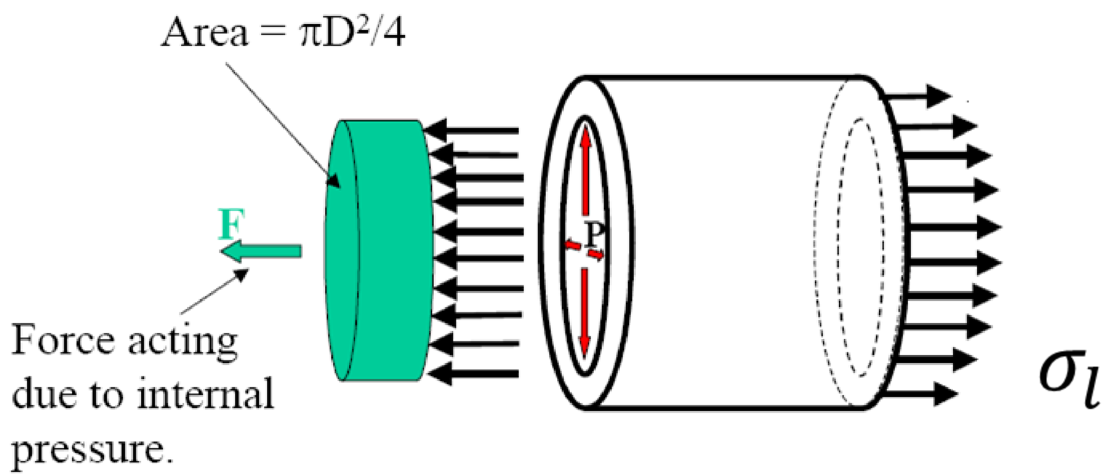


(a)

Fig.11.5. Longitudinal stress



(b)



(c)

Fig.11.5. Longitudinal stress

- The force tending to push the two halves apart is given by :

$$P = Intensity\ of\ internal\ pressure\ \times Area$$

$$= p \times \frac{\pi}{4}(d)^2$$

- The cross-section area of material which sustains this force is given by:

$$A = \pi dt$$

Therefore the longitudinal stress is given by:

$$\sigma_l = \frac{Total\ pressure}{Resisting\ section} = \frac{p \times \frac{\pi}{4}(d)^2}{\pi d t} = \frac{pd}{4t}$$

Note: If η is the efficiency of the riveted joints of the shell, then the stress, $\sigma_t = \frac{pd}{4t\eta}$

- **Note**
- Since hoop stress is twice the longitudinal stress, the cylinder would fail by tearing along a line parallel to the axis, rather than on a section perpendicular to the axis.
- The equation for hoop stress is therefore used to determine the cylinder thickness.
- Allowance is made for this by dividing the thickness obtained in hoop stress equation by efficiency (i.e. tearing and shearing efficiency) of the joint.

Ex:97. *A stream boiler of 800 mm diameter is made up of* 10 *mm thick plates. If the boiler is subjected to an internal pressure of* 2 *MPa, find the circumferential and longitudinal stresses induced in the boiler plates.*

Solution:

Given: Diameter of boiler *(d)* = 800 mm;

Thickness of plates *(t)* = 10 mm and

Internal pressure *(p)* = 2 MPa = 2 N/mm^2

- Circumferential stress induced in the boiler plates

$$\sigma_c = \frac{pd}{2t} = \frac{2 \times 800}{2 \times 10} = 80\ MPa$$

- Longitudinal stress induced in the boiler plates

$$\sigma_l = \frac{pd}{4t} = \frac{2 \times 800}{4 \times 10} = 40\ MPa$$

Ex:98. A cylindrical shell of 1.5 m diameter is made up of 16 mm thick plates. Find the circumferential and longitudinal stress in the plates, if the boiler is subjected to an internal pressure of 2.5 MPa. Take efficiency of the joints as 80%.

Solution: Given,

Diameter of shell *(d)* = 1.5 m = 1500 mm;

Thickness of plates *(t)* = 16mm;

Internal pressure *(P)* = 2.5 MPa

Efficiency (η)= 80% = 0.8

- Circumferential stress

$$\sigma_c = \frac{pd}{2t\eta} = \frac{2.5 \times (1.5 \times 10^3)}{2 \times 16 \times 0.8} = 146.48\ MPa$$

- Longitudinal stress

$$\sigma_l = \frac{pd}{4t\eta} = \frac{2.5 \times (1.5 \times 10^3)}{4 \times 16 \times 0.8} = 73.24\ MPa$$

Example 11.3: The diameter of a city water supply pipe is 750 mm. Its has to withstand a water head of 60 m. Find the thickness of the pipe, if the permissible stress is 18 N/mm^2. Take unit weight of water as 9810 N/m^3

Solution:

Pressure of water $= wh$

$$= 9810 \times 60\ N/m^2$$

$$= 588600\ N/m^2$$

$$= 0.5886 N/mm^2$$

The greatest stress is $\sigma_c = \frac{pd}{2t}$

Equating it to permissible stress(σ), we get

$$t = \frac{pd}{2\sigma} = \frac{0.5886 \times 750}{2 \times 18} = 12.26\ mm\ \ (\boldsymbol{Ans})$$

Ex:100. A cylindrical shell of 500 mm diameter is required to withstand an internal pressure of 4 MPa. Find the minimum thickness of the shell, if maximum tensile strength in the plate material is 300 MPa and efficiency of the joints is 70%. Take factor of safety as 5.

Soln:

Allowable tensile stress

(i.e., circumferential stress),

$$\sigma_c = \frac{Tensile\ strength}{Factor\ of\ safety} = \frac{300}{5} = 60\ MPa$$

and minimum thickness of shell,

$$t = \frac{pd}{2\sigma_c \eta} = \frac{4 \times 500}{2 \times 60 \times 0.7}$$

$$= 23.81\ mm\ (say\ 24\ mm)$$

CHANGE IN DIMENSIONS OF A THIN CYLINDRICAL SHELL

We know that the circumferential stress,

$$\sigma_c = \frac{pd}{2t}$$

and longitudinal stress,

$$\sigma_l = \frac{pd}{4t}$$

Now let

$$\delta d = change\ in\ diameter\ of\ the\ shell$$

$$\delta l = change\ in\ length\ of\ the\ shell\ and$$

$$\mu = Poisson's\ ratio$$

The state of stress on the shell wall is as shown in fig.11.6

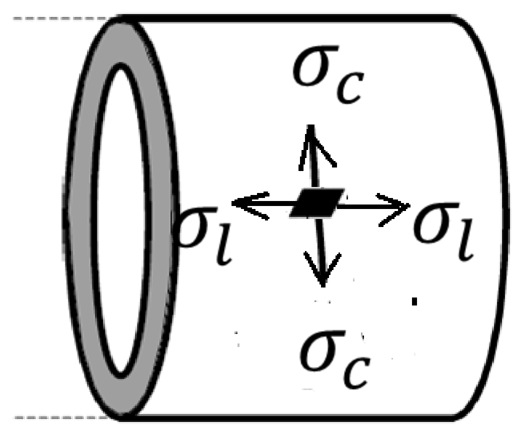


Fig.11.6.

Circumferential strain $e_c = e_1 = \frac{\sigma_c}{E} - \mu\left(\frac{\sigma_l}{E}\right)$

$$= \frac{pd}{2tE} - \mu\left(\frac{pd}{4tE}\right)$$

$$= \frac{pd}{4tE}\,(2-\mu)$$

$\therefore$ *change in diameter*, $\delta d = e_c\, d$

$$= \frac{pd^2}{4tE}\,(2-\mu)$$

- Longitudinal strain $e_l = e_2$

$$= \frac{\sigma_l}{E} - \mu\left(\frac{\sigma_c}{E}\right)$$

$$= \frac{pd}{4tE} - \mu\left(\frac{pd}{2tE}\right)$$

$$= \frac{pd}{4tE}\ (1-2\mu)$$

Change in length , δl = $e_l .\ l$

$$= \frac{pdl}{4tE}\ (1-2\mu)$$

We know ,

Initial volume of cylinder , V= $\frac{\pi\, d^2}{4}\, l$

∴ change in volume , δV = $\frac{\pi\, d^2}{4}\,\delta l$ + $\frac{\pi\, l}{4}$ 2d δd

∴ change in strain , e_v = $\frac{\delta v}{v}$ = $\frac{\delta l}{l}$ + 2 $\frac{\delta d}{d}$

= $e_l + 2\, e_d$

(or)

change in volume of cylinder shell:

Final volume = $(l + \delta l) \times \frac{\pi}{4}(d + \delta d)^2$

$$= \frac{\pi}{4}(l + \delta l) \times (d^2 + (\delta d)^2 + 2d\,\delta d)$$

$$= \frac{\pi}{4}\{\ d^2 l + l(\delta d)^2 + 2d\,\delta d\, l + d^2 \delta l + \delta l(\delta d)^2 + 2d\,\delta d \delta l\ \}$$

$$= \frac{\pi}{4}(d^2 l + 2\, l\, d\, \delta d + d^2 \delta l)$$

δv = final volume – initial volume

$$= \frac{\pi}{4}\left(d^2 l + 2l \text{ d } \delta\text{d} + d^2 \delta l\right) - \frac{\pi}{4} d^2 l$$

$$= \frac{\pi}{4}\left(2l\text{d } \delta\text{d} + d^2 \delta l \right)$$

$$\therefore\ e_v = \frac{\delta v}{v} = \frac{\frac{\pi}{4}\left(2l\text{d } \delta\text{d} + d^2 \delta l \right)}{\frac{\pi}{4} d^2 l}$$

$$= 2\,\frac{\delta d}{d} + \frac{\delta l}{l}$$

i.e., $e_v = \frac{pd}{4tE}\ (1-2\mu) + 2\ \frac{pd}{4tE}\ (2-\mu)$

$$= \frac{pd}{4tE}\ (5-4\mu)$$

$$\therefore\ change\ in\ volume =\ e_v\ v$$

$$= \frac{pd}{4tE}\ (5-4\mu)\,\frac{\pi\, d^2}{4}\, l$$

$$= \frac{pd^3\pi\, l}{16\ tE}\ (5-4\mu)$$

Example 11.5: A cylindrical shell is 3m long, and is having 1 m internal diameter and 15mm thickness. calculate the maximum intensity of shear stress induced and the changes in the shell, if it is subjected to an internal fluid pressure of 1.5N/mm^2

Take E = $2 \times 10^5 N/mm^2$ and $\mu = 0.3$

Solution:

$$l = 3m = 3000\, mm$$

$$d = 1m = 1000\, mm$$

$t = 15\, mm$ $\qquad$ $p = 1.5\ N/mm^2$

$E = 2 \times 10^5\ N/mm^2$ $\qquad$ $\mu = 0.3$

Hoop stress,

$$\sigma_c = f_1 = \frac{pd}{2t} = \frac{1.5 \times 1000}{2 \times 15} = 50 N/mm^2$$

Longitudinal stress,

$$\sigma_l = f_2 = \frac{pd}{4t} = \frac{1.5 \times 1000}{4 \times 15} = 25\ N/mm^2$$

$$q_{max} = \frac{f_1 - f_2}{2} = \frac{50 - 25}{2} = 12.5 N/mm^2$$

Now diametrical strain $\frac{\delta d}{d} = e_c = \frac{\sigma_c}{E} - \mu \frac{\sigma_l}{E}$

$$= \frac{1}{E}[\sigma_c - \mu\sigma_l]$$

$$= \frac{1}{E}[50 - 0.3 \times 25]$$

$$= 2.125 \times 10^{-4}$$

$$\delta d = 2.125 \times 10^{-4}\mathrm{d} = 2.125 \times 10^{-4} \times 1000$$

$$= 0.2125\, mm \qquad (\boldsymbol{Ans})$$

Longitudinal strain,

$$= \frac{\delta l}{l} = e_l = \frac{\sigma_l}{E} - \frac{\mu \sigma_c}{E}$$

$$= \frac{1}{E}[\sigma_l - \mu \sigma_c]$$

$$= \frac{1}{E}[25 - 0.3 \times 50]$$

$$= 5 \times 10^{-5}$$

$$\delta l = 5 \times 10^{-5} \times l = 5 \times 10^{-5} \times 3000$$

$$= 0.15\, mm \qquad (\boldsymbol{Ans})$$

Change in Volume ,

$$\frac{\delta V}{V} = 2e_c + e_l = 2 \times 2.125 \times 10^{-5} + 5 \times 10^{-5}$$

$$= 4.75 \times 10^{-4}$$

$$\delta V = 4.75 \times 10^{-4} \times \frac{\pi}{4} 1000^2 \times 3000$$

$$= 1119192.4 \; mm^3$$

(Ans)

Example 11.6: A thin cylindrical shell, 2m long has 200 mm diameter and thickness of metal 10 mm. It is filled completely with a fluid atmospheric pressure. If an additional 25 000 mm^3 fluid is pumped in, find the pressure developed and hoop stress developed. Find also the changes in diameter and length.

Take E = $2 \times 10^5 N/mm^2$ and $\mu = 0.3$

Solution:

Let the pressure developed be ‘p’. Then circumferential stress

$$\sigma_c = \frac{pd}{2t} = \frac{p \times 200}{2 \times 10} = 10p$$

Longitudinal stress,

$$\sigma_l = \frac{pd}{4t} = \frac{p \times 1000}{4 \times 10} = 5p$$

Diametrical strain = Circumferential strain

$$e_d = \frac{\delta d}{d} = \frac{\sigma_c}{E} - \frac{\mu \sigma_l}{E}$$

$$= \frac{1}{E}[10p - 0.3 \times 5p]$$

$$e_d = \frac{8.5p}{E}$$

Longitudinal strain e_l is given by

$$\frac{\delta l}{l} = e_l = \frac{1}{E}[\sigma_l - \mu\sigma_c]$$

$$= \frac{1}{E}[5p - 0.3 \times 10p]$$

$$= \frac{2p}{E}$$

Volumetric strain: $\frac{\delta V}{V} = 2e_d + e_l$

$$= 2 \times 8.5\,\frac{p}{E} + 2\,\frac{p}{E}$$

$$= \frac{19p}{E}$$

∴ Pressure developed,

$$p = \frac{\delta V}{V} \times \frac{E}{19}$$

$$= \frac{25000}{\frac{\pi}{4} \times 200^2 \times 2000} \times \frac{2 \times 10^5}{19}$$

$$= 4.188\ N/mm^2$$ **(Ans)**

$$\text{Hoop stress} = \frac{pd}{2t} = \frac{4.188 \times 200}{2 \times 10}$$

$$= 41.88\ N/mm^2 \qquad \textbf{(Ans)}$$

$$Diametrical\ strain, \frac{\delta d}{d} = e_d = \frac{8.5p}{2 \times 10^5}$$

$$\therefore\ change\ in\ diameter, \delta d = \frac{8.5 \times 4.188}{2 \times 10^5} \times 200$$

$$= 0.0356mm\ \textbf{(Ans)}$$

$$Longitudinal\ strain, \frac{\delta l}{l} = e_l = \frac{2p}{2 \times 10^5}$$

$$\therefore\ change\ in\ length, \delta l = \frac{2p}{E} \times l$$

$$= \frac{2 \times 4.188}{2 \times 10^5} \times 2000 = 0.08376\ mm$$ **(Ans)**

Ex:103. A cylindrical boiler is subjected to an internal pressure, p. If the boiler has a mean diameter, d and a wall thickness, t, derive expressions for the hoop and longitudinal stresses in its wall. If Poisson's ratio for the material is 0.30, find the ratio of the hoop strain to the longitudinal strain and compare it with the ratio of stresses.

Hoop stress (σ_c) will cause expansion on the lateral direction and is equal to $\sigma_y, while\ the\ longitudinal\ stress(\sigma_l)\ is\ \sigma_x$

Hoop stress, $\sigma_c = \frac{pd}{2t} = \sigma_y$

Longitudinal stress , $\sigma_l = \frac{pd}{4t} = \sigma_x$

Stress ratio $= \frac{hoop\ stress}{longitudinal\ stress} = 2$

$$hoop\ strain, e_y = \frac{1}{E}\left(\sigma_y - \mu\sigma_x\right)$$

$$= \frac{1}{E}\left(\frac{pd}{2t} - 0.3\,\frac{pd}{4t}\right) = \frac{0.425}{E}\ \frac{pd}{t}$$

$$longitudinal\ strain,\quad e_x = \frac{1}{E}\left(\sigma_x - \mu\sigma_y\right)$$

$$= \frac{1}{E}\left(\frac{pd}{4t} - 0.3\,\frac{pd}{2t}\right) = \frac{0.1}{E}\ \frac{pd}{t}$$

$$\therefore \text{Strain ratio} = \frac{hoop\ strain}{longitudinal\ strain} = \frac{0.425}{0.1} = 4.25$$

WIRE-BOUND THIN CYLINDRICAL SHELLS

- Sometimes, we have to strengthen a cylindrical shell against bursting in longitudinal section*(i.e.,* due to hoop or circumferential stress).
- This is done by winding a wire under tension, closely round the shell as shown in Fig. 11.7.
- Its effect will be to put the cylinder wall under initial compressive stress.

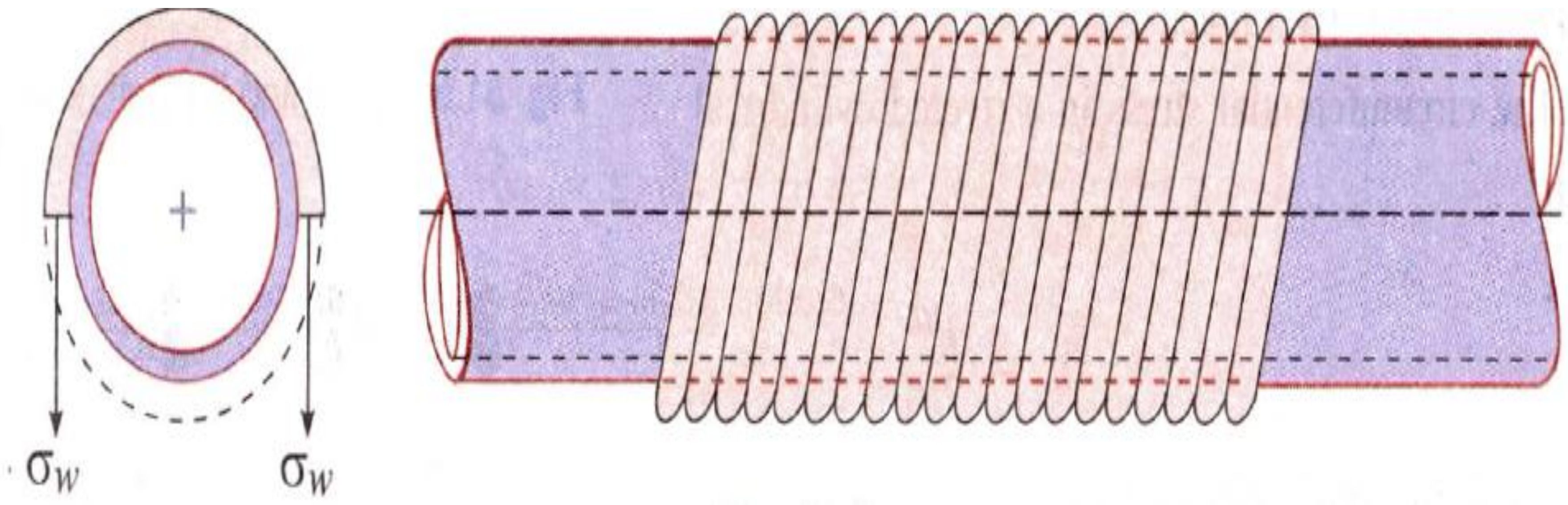


Fig. 11.7.

The tension in wires gives rise to external pressure on the cylinder and hence introduces compressive hoop stress in the cylinder.

- When internal pressure is applied, hoop tension is introduced.
- Hence, net hoop stress is algebraic sum of the above two types of stresses, which is obviously less than hoop tension that would have developed if there is no wire winding.
- This technique of strengthening cylinders is usually adopted for materials which are not very strong in tension (like cast iron) and in such cases steel wires are used.

Let, d' - the diameter of steel wire

d - diameter of the cylinder

t - thickness of cylinder

l- length of cylinder

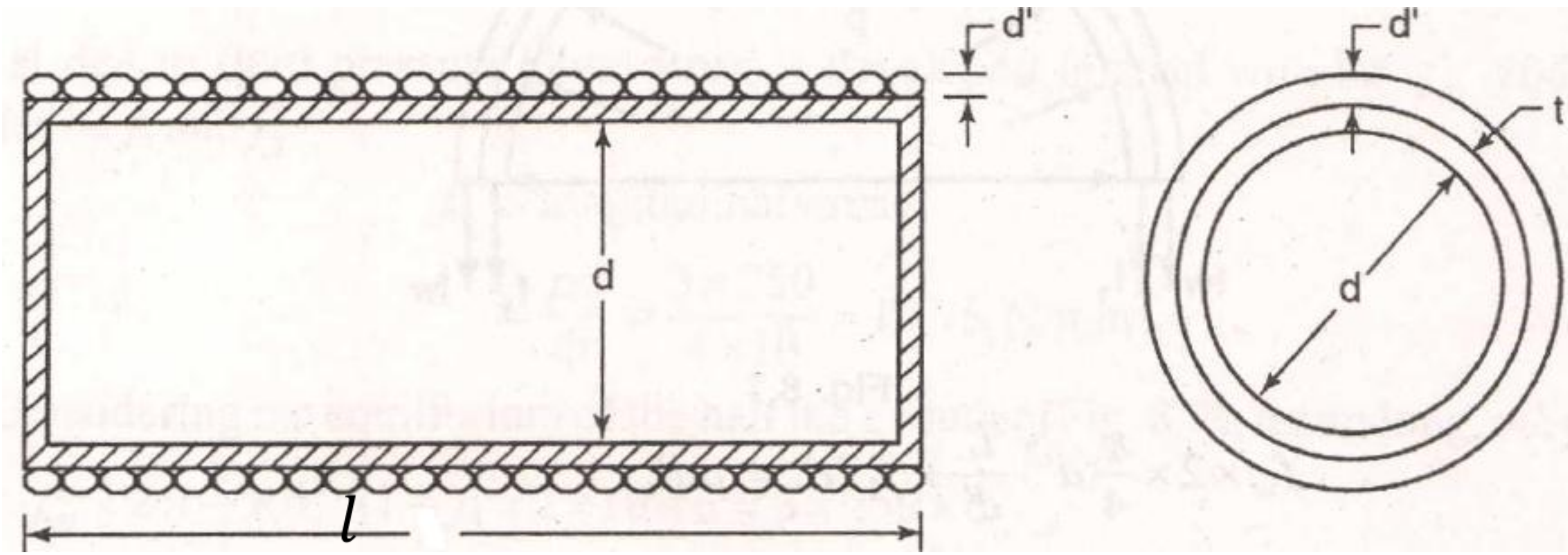


Fig. 11.8.

σ_{wo} - Initial tensile stress in steel wire

σ_e –Initial compressive stress in cylinder

To find the initial stresses, consider the equilibrium of half the cylinder (Fig. 11.9)

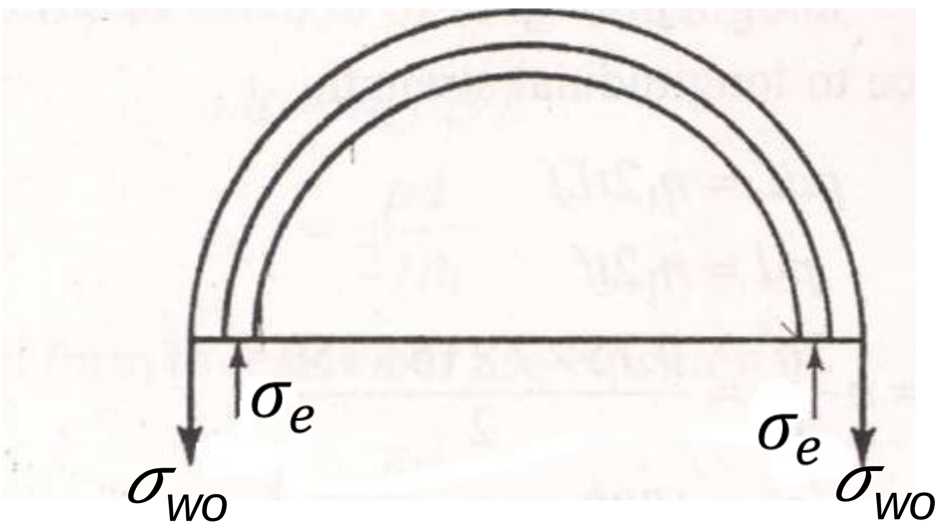


Fig. 11.9.

Force exerted by one turn of wire = $2\times \frac{\pi}{4}\,(d')^2\sigma_{wo}$

No. of turns in length l of the cylinder = $\dfrac{l}{d'}$

Total force exerted by wires

$$= \frac{l}{d'} \times 2 \times \frac{\pi}{4}\,(d')^2\sigma_{wo} \qquad \text{-----------(a)}$$

This is resisted by compressive stresses developed in cylinder.

Resisting force = $\sigma_e \times 2tl$ ----------(b)

Equating *(a)* and *(b)* for equilibrium, we get

$$\sigma_e \times 2tl = \frac{l}{d'} \times 2 \times \frac{\pi}{4} d'^2 \sigma_{wo}$$

$$\sigma_e = \frac{\pi}{4t} d'^2 \sigma_{wo} \quad \text{----------(1)}$$

When fluid is admitted:

Let the stress developed due to fluid

. . . . . .

Considering the forces acting on half the cylinder (Fig. 11.10) the equilibrium equation can be written as

$$\text{pd}l = \left(\sigma_w \times 2 \times \frac{\pi}{4}\text{d'}^2 \times \frac{l}{d\prime}\right) + (\sigma_c \times 2tl)$$

$$\sigma_w \frac{\pi}{4} d' + 2\sigma_c t = \text{pd} \qquad \text{----------(c)}$$

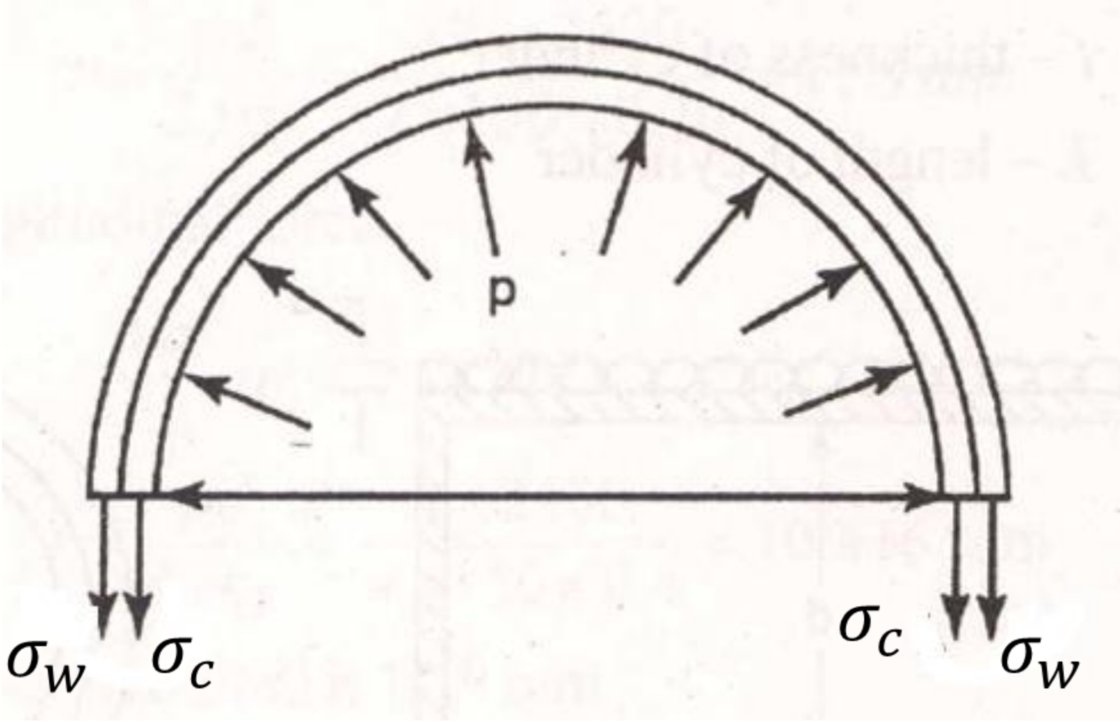


Fig. 11.10.

To form another equation in σ_w and σ_c, the criteria used is at common surface,

strain in wire and cylinder are the same.

- Now at common surface

$$\text{strain in wire} = \frac{\sigma_w}{E_w} \qquad \text{------(d)}$$

$$\text{Circumferential strain in cylinder} = \frac{\sigma_c}{E} - \mu\frac{\sigma_l}{E} \quad \text{----- (e)}$$

Where $\sigma_l - longitudinal\ stress = \frac{pd}{4t}$

- Equating (d) and (e) for compatibility , we get
- $\frac{\sigma_w}{E_w} = \frac{\sigma_c}{E} - \mu \frac{\sigma_l}{E} = \frac{\sigma_c}{E} - \mu \frac{pd}{4tE}$ ----------(f)
- From equation *(c)* and *(f),* stresses in wire σ_w , and in cylinder σ_c due to fluid alone can be found.
- Final stresses are obtained by algebraically adding initial stresses and stresses due to fluid

Example 11.8: A 250 mm diameter cast iron pipe has metal thickness of 10 mm. it is closely wound with 6mm diameter steel wire with an initial stress of 80 N/mm^2. Find the final stress developed in cylinder and wire when fluid is admitted at a pressure of $3N/mm^2$ Take $E_c = 100kN/mm^2$,

$\mu = 0.3$ and ,

$E_c = 200kN/mm^2$

Solution:

Consider 6 mm length of cylinder.

No of wires = 1

Force exerted by steel wire at diametrical section

$$= \sigma_{w0} \times 2 \times \frac{\pi d'^2}{4} \times 1$$

$$= 80 \times 2 \times \frac{\pi \times 6^2 \times 1}{4} = 4523.89\ N$$

If initial stress is σ_e in cylinder, then

$\sigma_e \times 2t \times 6 = 4523.89$ N

$$\sigma_e = \frac{4523.89}{2 \times 10 \times 6} = 37.7 N/mm^2$$

Let due to fluid pressure alone stresses developed in steel wire be σ_ω and in cylinder be σ_c and σ_l

$\sigma_l = longitudinal\ stress$

$$\sigma_l = \frac{pd}{4t} = \frac{3 \times 250}{4 \times 10} = 18.75\ N/mm^2$$

Considering the equilibrium of the half the cylinder (Figure 11.10), 6mm long, we get

$$\sigma_\omega \times 2 \times \frac{\pi}{4} \times 6^2 \times 1 + \sigma_c \times 2 \times 10 \times 6$$

$$= 3 \times 250 \times 6$$

$$56.55\sigma_\omega + 120\sigma_c = 4500$$

$$\sigma_\omega + 2.122\sigma_c = 79.58 ------(a)$$

Equating strain in wire to circumferential strain in cylinder, we get

$$\frac{\sigma_\omega}{E_s} = (\sigma_c - \mu\sigma_l)\frac{1}{E_c}$$

Now, $E_w = 200kN/mm^2 \;= 2 \times 10^5 N/mm^2$

$$E_s = 100kN/mm^2 \;= 1 \times 10^5 N/mm^2$$

$$\therefore \qquad \frac{\sigma_\omega}{2\times10^5} = (\sigma_c - 0.3 \times 18.75)\frac{1}{1\times10^5}$$

$$\sigma_\omega = 2\sigma_c - 11.25 ------(b)$$

Substituting it in (a) we get

$$2\sigma_c - 11.25 + 2.122\sigma_c = 79.58$$

$$\sigma_c = \frac{79.58 + 11.25}{4.122}$$

$$= 22.035\, N/mm^2$$ **(tensile)**

$\therefore$ from (b) $\quad \sigma_\omega = 32.821\, N/mm^2$ **(tensile)**

Final stresses are

(a)In steel wire $= 80 + 32.821$

$$= 112.821\, N/mm^2$$ **(Ans)**

(a)In cylinder $= -37.7 + 22.035$

$$= -15.664\, N/mm^2$$ **(Ans)**

Ex:105. *A cylinder made of bronze 180 mm outside diameter and 15 mm thick* is *strengthened by a single layer of steel wire* 2·25 *mm diameter wound over it under a constant stress of 75 MN/m²^2. The cylinder* is *subjected to an internal pressure of 27 MN/m^2 with rise in temperature of the cylinder by 120°c. Assuming the cylinder to be a thin shell with closed ends.*

determine the final values of

(i) *The stress in the wire ;*

(ii) *The circumferential stress in the cylinder wall ;*

(iii) *The radial pressure between the wire and the cylinder.*

*Take: For steel: E = 208 GN/m², *

$$\alpha = 11.18 \times 10^{-6} \, per \, °C$$

For brass: E = 90 GN/m²,

$$\alpha = 18.6 \times 10^{-6} \, per \, °C$$

Poisson's ratio = 0·32

(i) Stress in the wire

<u>Before subjecting the cylinder to internal pressure;</u>

Tensile force extended by wire per unit length = compressive force developed in the cylinder

i.e. $2\times \frac{\pi}{4} d'^2 \times \sigma_{w0} \times n = 2t \times l \times \sigma_e$

(or)

$2\times \frac{\pi}{4} d'^2 \times \sigma_{w0} \times \frac{1}{d\prime} = 2t.\, \sigma_e$

$$\therefore \sigma_e = \frac{\pi . d'}{4t} . \sigma_{w0}$$

Where σ_e= initial circumferential compressive stress in the cylinder

$$= \frac{\pi(2.25)}{4 \times 15} \times 75 = 8.84\ N/mm^2$$

<u>After subjecting the cylinder to internal pressure</u>

Due to internal pressure , the longitudinal stress developed in the cylinder ,

$$\sigma_l = \frac{pd}{2t} = \frac{27 \times 150}{4 \times 15} = 67.5\ N/mm^2$$

For equilibrium,

Total bursting force = total resisting force

(per unit length)

$$p \times d \times l = (\sigma_c \times 2t \times l) + (\sigma_w \times 2 \times \frac{\pi}{4}(d')^2 \times n$$

$$\therefore pd = \sigma_c \times 2t + \sigma_w \times \frac{\pi}{2} d'$$

$$\therefore n = \frac{1}{d'}$$

$$27\times 150 = (\sigma_c \times 2 \times 15) + \left(\sigma_w \times \pi \times \frac{2.25}{2}\right)$$

$$\sigma_c + 0.1178\sigma_w = 135 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Equating the strain in cylinder and wire , we have

$$\frac{\sigma_c}{E_c} - \mu\frac{\sigma_l}{E_c} + \propto_b . T = \frac{\sigma_w}{E_w} + \propto_s . T$$

$$\left(\frac{\sigma_c}{90 \times 10^3} - 0.32 \times \frac{67.5}{90 \times 10^3}\right) + 18.6 \times 10^{-6} \times 120$$

$$= \frac{\sigma_w}{208 \times 10^3} + 11.8 \times 10^{-6} \times 120$$

$$\sigma_c - 21.6 + 200.88 = 0.433\sigma_w + 127.44$$

$$\therefore \sigma_c = 0.433\sigma_w - 57.84 \ldots\ldots\ldots\ldots\ldots..(2)$$

Solving (1) and (2)

$$\sigma_w = 340.9\ N/mm^2$$

$$\sigma_c = 95.46\ N/mm^2$$

Final $stress\ in\ the\ wire$= $\sigma_{w0} + \sigma_w$

= 7.5 + 340.9 = 415.9 N/mm^2

(ii) The circumferential stress in the cylinder wall

Final stress in the cylinder = $\sigma_e + \sigma_c$

=(-8.84)+ 95.46 = 86.62 N/mm^2

(iii) The radial pressure between the wire and the cylinder.(σ_r)

The radial pressure is due to wire winding and temperature rise.

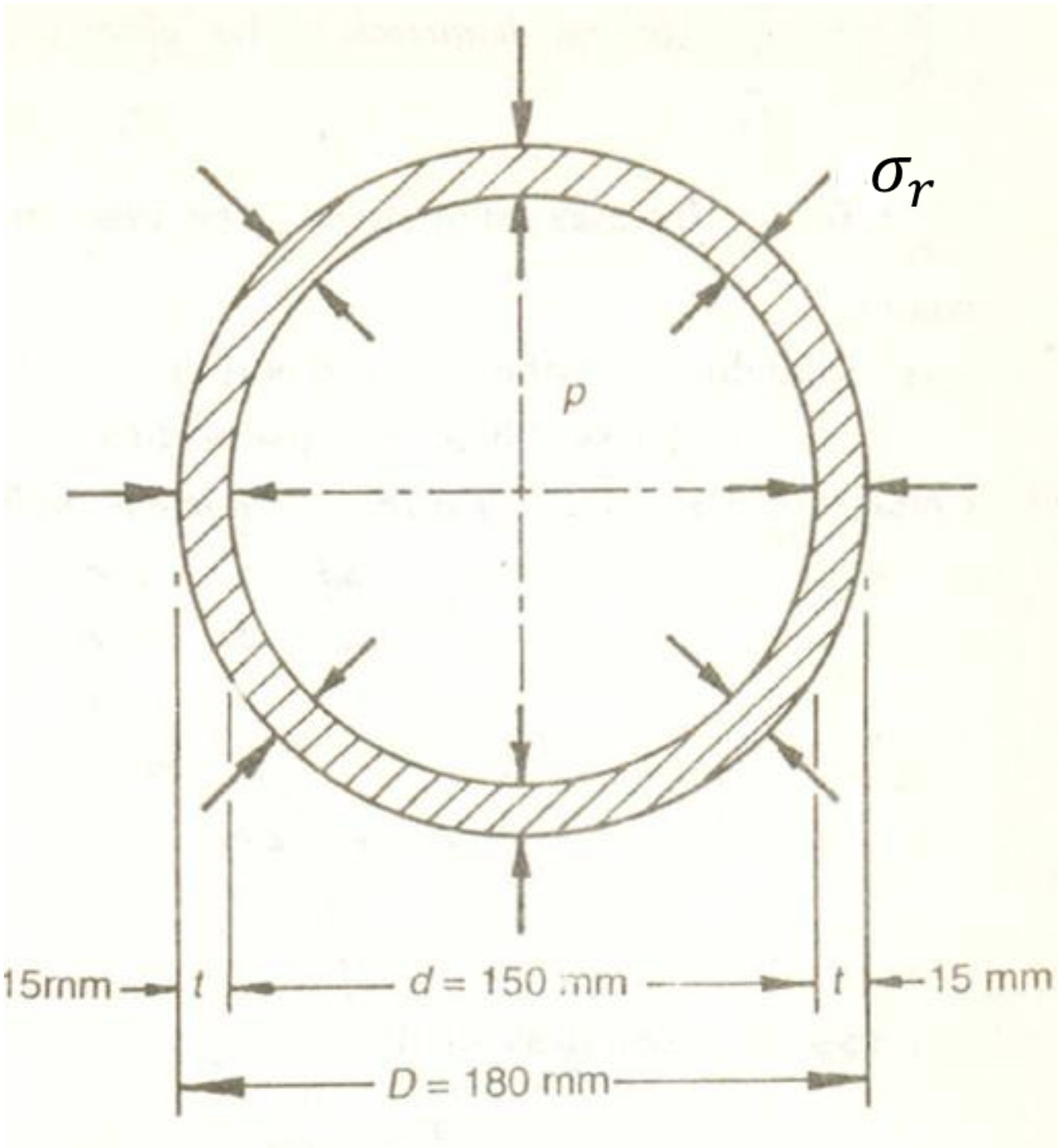


Fig. 11.11

Final circumferential stress $= \frac{pd}{2t} - \frac{\sigma_{rD}}{2t}$

i.e.,

$$86.62 = \frac{27 \times 150}{2 \times 15} - \frac{\sigma_r \times 180}{2 \times 15}$$

$86.62 = 135 - 6\sigma_r$

$$\therefore \sigma_r = 8.06 N/mm^2$$

THIN SPHERICAL SHELLS

- Consider a thin spherical shell subjected to an internal pressure as shown in Fig. 11.12
-

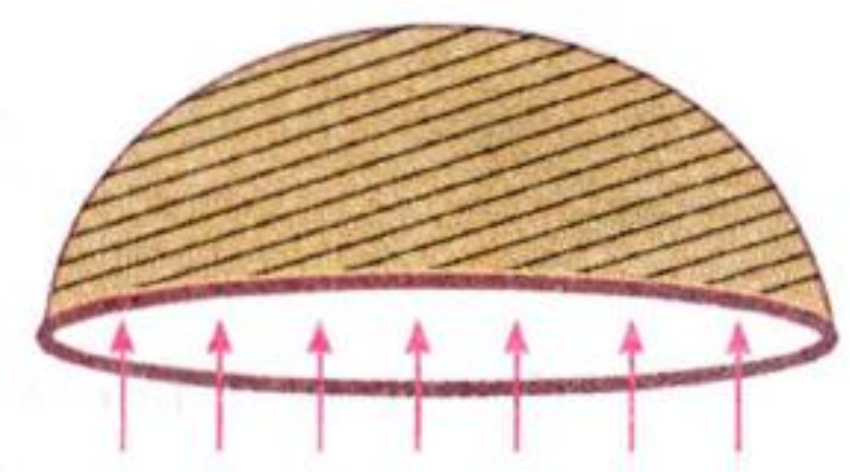


Fig. 11.12 Spherical shell

Let p = Intensity of internal pressure,

d = Diameter of the shell and

t = Thickness of the shell,

The shell is likely to be torn away along the centre of the sphere.

Therefore, total pressure acting along the centre of the sphere,

$$P = \text{intensity of internal pressure} \times area$$

$$= p \times \frac{\pi}{4} \times d^2$$

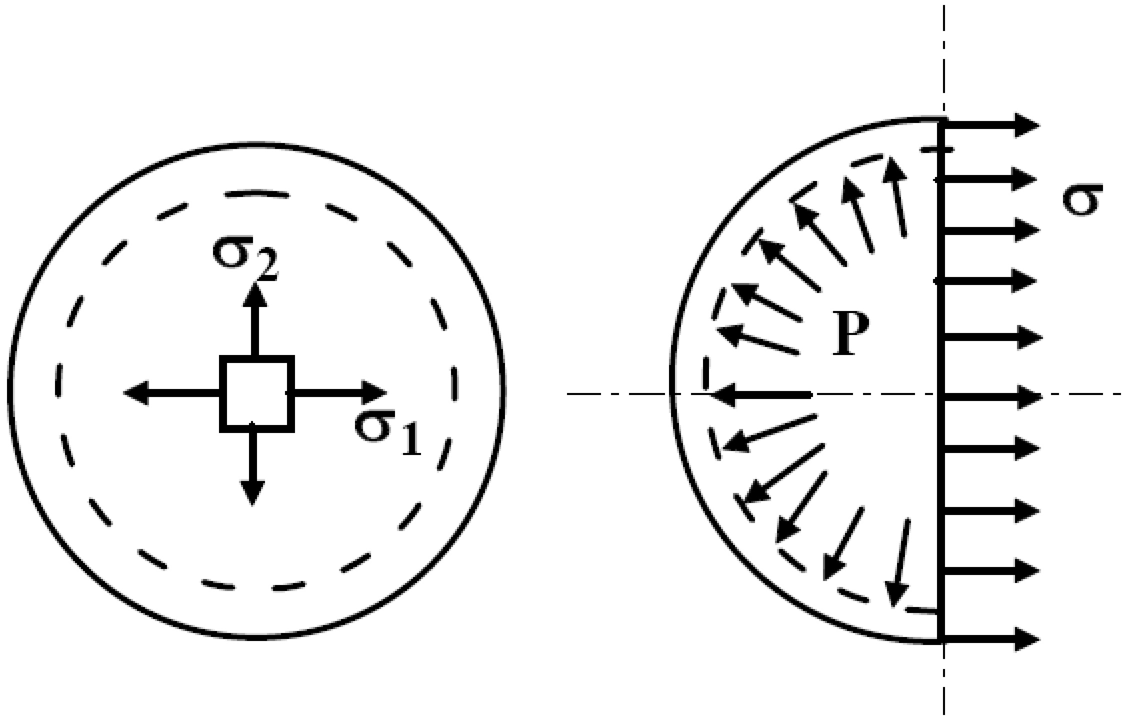


Fig. 11.13

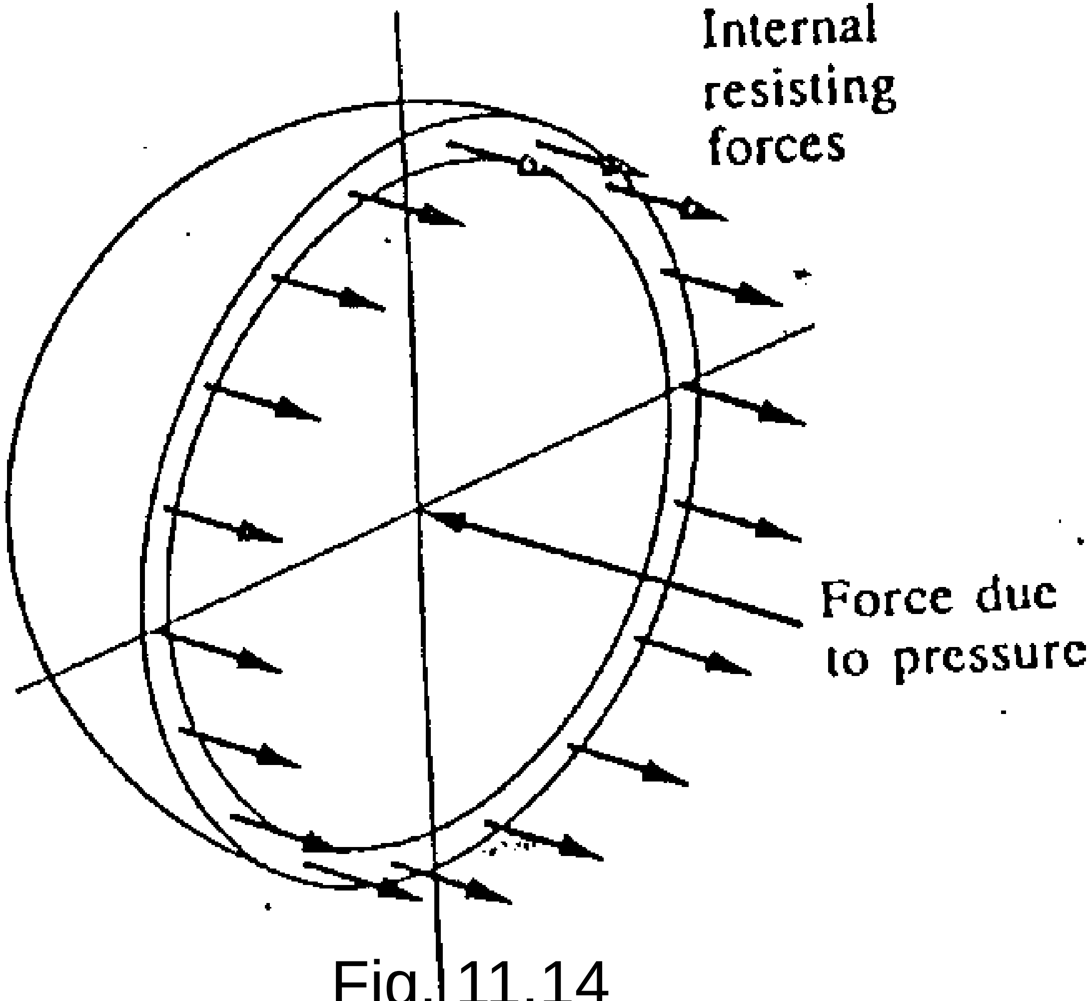


Fig. 11.14

stress in the shell material,

$$\sigma = \frac{total\ pressure}{resisting\ pressure} = \frac{p \times \frac{\pi}{4} \times d^2}{\pi d \times t} = \frac{pd}{4t}$$

$NOTE;$

$If, \eta\ is\ the\ efficiency\ of\ the\ riveted\ joints\ of\ the$

$spherical\ shell, then\ stess$

- $\sigma = \frac{pd}{4t\eta}$

CHANGE IN DIAMETER AND VOLUME OF A THIN SPHERICAL SHELL

We know that

$$\sigma_c = \frac{pd}{4t}\ , \sigma_l = \frac{pd}{4t}$$

Circumferential strain

$$e_c = e_l\ ,\ e_1 = e_2$$

$$= \frac{\sigma_c}{E} - \mu\ \frac{\sigma_l}{E} = \frac{pd}{4tE}\ (1 - \mu)$$

Change in diameter $\delta d = e_d$.d

$$= \frac{p\ d^2}{4tE}\ (1 - \mu)$$

Volume of sphere

$$V = \frac{4}{3}\pi\ r^3$$

$$= \frac{4}{3}\pi\ (\frac{d}{2})^3$$

$$= \frac{\pi\ d^3}{6}$$

$\therefore$ change in volume , $\delta v = \frac{\pi}{6} 3\ d^2\ \delta d$

$$\therefore volumetric\ strain, e_v = \frac{\delta v}{v} = \frac{3\delta d}{d}$$

$$= 3\ .e_d$$

(or)

We know, Initial volume of sphere , v = $\frac{\pi}{6}\ d^3$

Final volume , $\tilde{V} = \frac{\pi}{6}(d + \delta d)^{\ 3}$

$$= \frac{\pi}{6}(\ d^3 + 3d^2\delta d + 3d(\delta d^2) + (\delta d)^3)$$

by neglecting product of two or more small quantities.

$$\tilde{V} = \frac{\pi}{6}\left(d^3 + 3\, d^2 \delta d \right)$$

$$\therefore \textit{change in volume } \delta v = \tilde{V} - V$$

$$= \frac{\pi}{2}\, d^2 \delta d$$

$$\therefore \textit{volumetric strain}, e_v = \frac{\delta v}{v} \qquad = \frac{\frac{\pi}{2} d^2 \delta d}{\frac{\pi}{6} d^3}$$

$$= 3\,\frac{\delta d}{d} \qquad = 3.\ e_d$$

$$\therefore e_v = 3 \times \frac{p\,d^2}{4tE}\,(1-\mu)$$

$and\ change\ in\ volume$, $\delta v = e_v \times$ volume

$$= 3 \times \frac{p\,d}{4tE}\,(1-\mu) \times \frac{\pi}{6} d^3$$

$$= \frac{p\pi\,d^4}{8tE}\,(1-\mu)$$

Example 11.10 At atmospheric pressure, a thin spherical shell has diameter 750 mm and thickness 8 mm. Find the stress introduced and change in diameter and volume when the fluid pressure is increased to 2.5 N/mm^2.

Take E = $2 \times 10^5 N/mm^2$ and $\mu = 0.3$

Solution:

$$d = 750\,mm, t = 8mm\ and\ p = 2.5N/mm^2$$

Hoop stress $= f_1 = f_2 = \frac{pd}{4t}$

$$= \frac{2.5 \times 750}{4 \times 8} = 58.59N/mm^2$$

$$\frac{\delta d}{d} = \frac{pd}{4tE}[1 - \mu]$$

$$\frac{\delta d}{750} = \frac{2.5 \times 750}{4 \times 8 \times 2 \times 10^5}[1 - 0.25]$$

$$\delta d = 0.22\,mm$$

(Ans)

$$\frac{\delta V}{V} = 3\,\frac{pd}{4tE}\,[1 - \mu]$$

$$\delta V = \frac{3 \times 2.5 \times 750(1 - 0.25)}{4 \times 8 \times 2 \times 10^5} \times \frac{\pi}{6} \times 750^3$$

$$= 145608.33\; mm^3$$ **(Ans)**

CYLINDRICAL SHELL WITH HEMISPHERICAL ENDS

Fig. 11.15 shows a thin cylindrical shell with spherical ends.

Let d = internal diameter of the cylinder

t_1 = wall thickness of cylindrical portion

t_2 = wall thickness of hemispherical portion

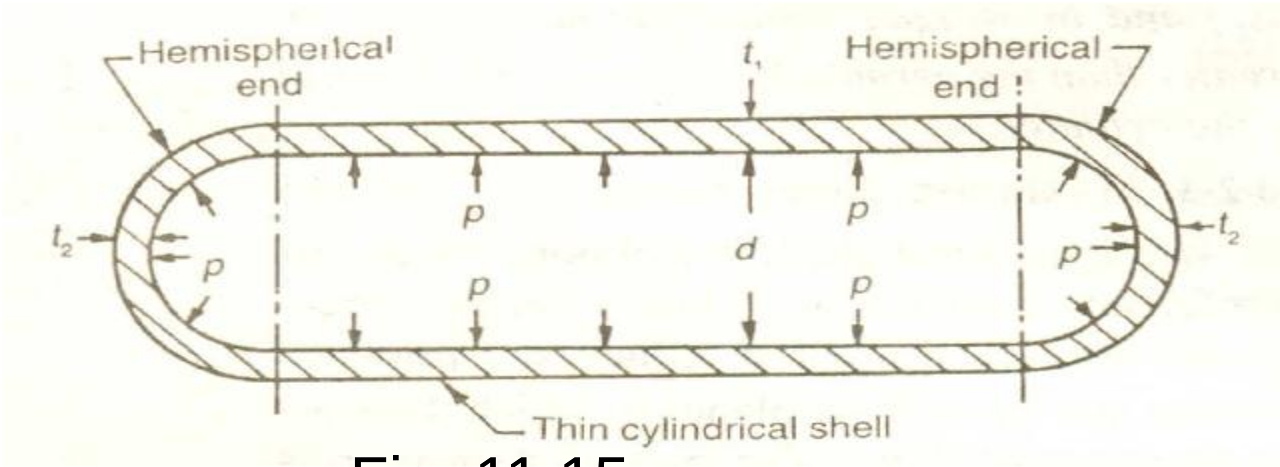


Fig. 11.15

Circumferential stress developed in cylindrical portion

$$\sigma_{c_1} = \frac{pd}{2t_1}$$

longitudinal stress developed in cylindrical portion $\sigma_{l_1} = \frac{pd}{4t_1}$

Circumferential strain developed in cylindrical portion, $e_{c_1} = \frac{\sigma_{c_1}}{E} - \mu \frac{\sigma_{l_1}}{E}$

$$= \frac{pd}{4t_1 E}(2 - \mu)$$

Circumferential stress developed in hemispherical portion,

$$\sigma_{C_2} = \frac{pd}{4t_2}$$

Circumferential strain in hemispherical portion,

$$e_{c_2} = \frac{\sigma_{c_2}}{E} - \mu\frac{\sigma_{l_2}}{E}$$

$$= \frac{pd}{4t_2E}(1-\mu)$$

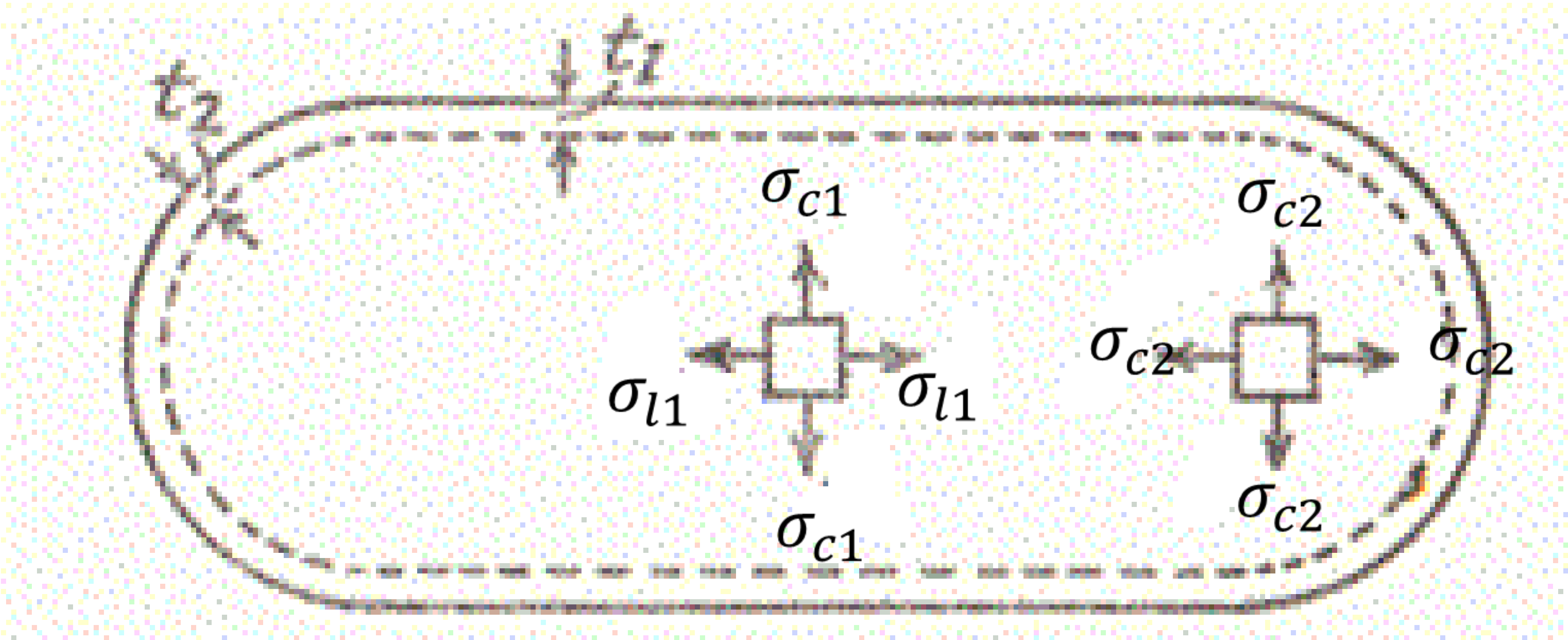

t_1
t_2
σ_{c1}
σ_{c2}
σ_{c2}
σ_{c2}
σ_{l1}
σ_{l1}
σ_{c2}
σ_{c1}

Fig. 11.16

- In order that there is *no distortion* at the junction of cylindrical and hemispherical portions the circumferential strains in the two have to be *equal.*

i.e., $e_{c_1} = e_{c_2}$

$$\frac{pd}{4t_1 E}(2-\mu) = \frac{pd}{4t_2 E}(1-\mu)$$

$$\therefore \frac{t_2}{t_1} = \frac{1-\mu}{2-\mu}$$

- Since *(1-μ)* is always less than *(2 - μ)*, whatever be the value of μ, thus t_2 shall always be less than t_1.
- *i.e., the hemispherical end is always thinner than the cylindrical portion.*
- So that the maximum stress may be *same* in both cylindrical and hemispherical portions, we have $\sigma_{c_1} = \sigma_{c_2}$
- $\frac{pd}{2t_1} = \frac{pd}{4t_2}$ (or) $\frac{t_2}{t_1} = 0.5$

Example.11.11 The cylindrical shell shown in figure 11.17 is 1m long and has diameter 200 mm.The metal thickness is 6mm.It has spherical ends. Calculate the change in volume, if internal pressure is raised to 1.5 N/mm^2. E = $2 \times 10^5 N/mm^2$ and $\mu = 0.25$

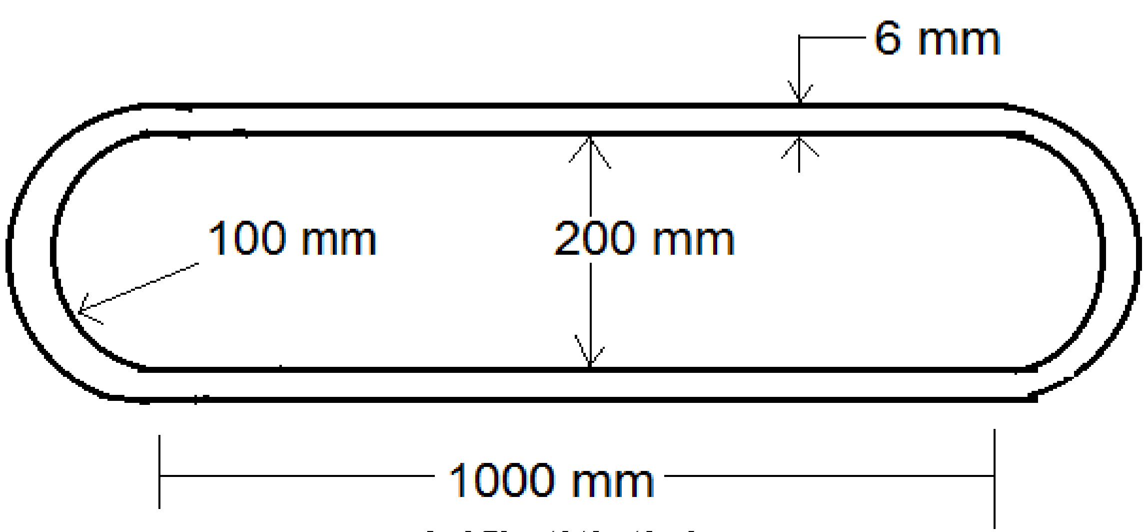


Fig. 11.17

Solution:

Change in volume of sphere:

Change in volume in two hemispheres = Change in volume in a sphere

Now,

$$\frac{\delta V}{V} = \frac{3pd}{4tE}[1-\mu]$$

$$\delta V_s = \frac{3 \times 1.5 \times 200}{4 \times 6 \times 2 \times 10^5}[1-0.25] \times V_s$$

$$= \frac{3\times1.5\times200}{4\times6\times2\times10^5}[0.75] \times \frac{\pi}{6} \times 200^3$$

$$= 589.05\ mm^3$$

Change in volume of cylinder:

$$\sigma_c = \frac{pd}{2t} = \frac{1.5 \times 200}{2 \times 6} = 25N/mm^2$$

$$\sigma_l = \frac{pd}{4t} = \frac{1.5 \times 200}{4 \times 6} = 12.5N/mm^2$$

$$e_c = \frac{\sigma_c}{E} - \frac{\mu\sigma_l}{E} = \frac{25}{E} - \frac{0.25 \times 12.5}{E}$$

$$e_c = \frac{21.875}{E}$$

$$e_l = \frac{\sigma_l}{E} - \frac{\mu\sigma_c}{E} = \frac{12.5}{E} - \frac{0.25 \times 25}{E}$$

$$= \frac{6.25}{E}$$

$$\frac{\delta V_c}{V_c} = 2e_c + e_l$$

$$= \frac{2 \times 21.875 + 6.25}{E} = \frac{50}{E}$$

$$\delta V_c = \frac{50}{V_c} \times V_c = \frac{50}{2 \times 10^5} \times \frac{\pi}{4} \times 200^2 \times 1000$$

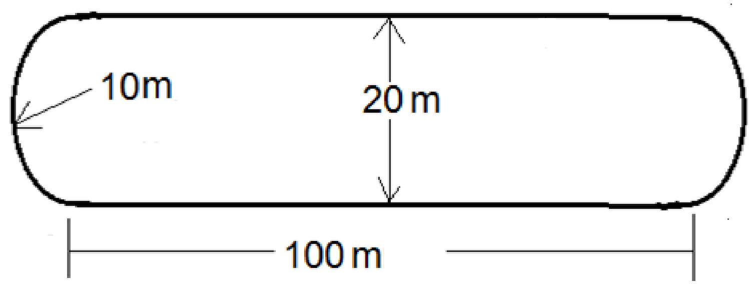
$$= 7853.98\ mm^3$$

Total change in volume $= \delta V_s + \delta V_c$

$$= 589.05 + 7853.98$$

$= 8443.03 mm^3$ **(Ans)**

Ex:108. The dimensions of an oil storage tank with hemispherical ends are shown in the Fig. The tank is filled with oil and the volume of oil increases by 0.1% for each degree rise in temperature of 1^0C. If the coefficient of linear expansion of the tank material is 12×10^{-6} per ^{0}C, how much oil will be lost if the temperature rises by 10^0C



For 10 ° rise in temperature;

Volumetric strain of oil = $0.001 \times 10 = 0.01$

Volumetric strain of tank = $3 \propto T$

$= 3 \times 12 \times 12^{-6} \times 10 = 0.00036$

Difference in volumetric strain = 0.01 – 0.00036

= 0.00964

For volume of tank

$$= (\pi \times 10^2 \times 100) + (\frac{4}{3} \times \pi \times 10^3)$$

$$=10000\,\pi + 1333.33\,\pi = 11333.33\pi\ m^3$$

Volume of oil lost

$$= \text{strain difference} \times volume\ of\ tank$$

$$= 0.00984 \times 11333.33\,\pi$$

$$= 343.2\ m^3$$

BIAXIAL STATE OF STRESS

So far we have considered stresses arising in bars subjected to axial loading, beams subjected to bending and shafts subjected to torsion, as well as several cases involving thin walled pressure vessels.

It is to be noted that we have considered a member, for example, to be subjected to only one loading at a time, such as bending.

Biaxial state of stress ...

But frequently members are simultaneously subjected to several of the previously mentioned loading, and it is required to determine the state of stress under these conditions.

- The wall of a thin walled cylinder is in a biaxial state of stress (hoop stress in circumferential direction and longitudinal stress in length direction)

- The wall of a thin walled sphere is in a biaxial state of stress (hoop stress in two orthogonal directions)

- An element in a web of beam has normal stress and shear stress due to bending or loading.

- A bar subjected to axial load and twisting, the material has normal stress due to axial load and shear stress due to torsion.

TRANSFORMATION OF PLANE STRESSES

- Though the state of stress at a point in a stressed body remains the same, the normal and shear stress components vary as the orientation of plane through that point changes.
- Under complex loading, a structural member may experience larger stresses on inclined planes than on the cross section.

Transformation of plane stresses….

- The knowledge of maximum normal and shear stresses and their plane's orientation assumes significance from failure point of view.
- The material will fail (or crack) along the plane on which the normal stress is maximum.
- Hence, it is important to know how to transform the stress components from one set of coordinate axes to another set of co-ordinates axes.

Expression for Normal and tangential stress at a point on any plane in a strained body subjected to Biaxial stresses combined with shear stresses

Consider a rectangular body ABCD of unit thickness subjected to direct stresses f_x and f_y and shear stresses q as shown in Fig.12.1(a).

Let RS be the oblique plane making an angle θ with the vertical plane.

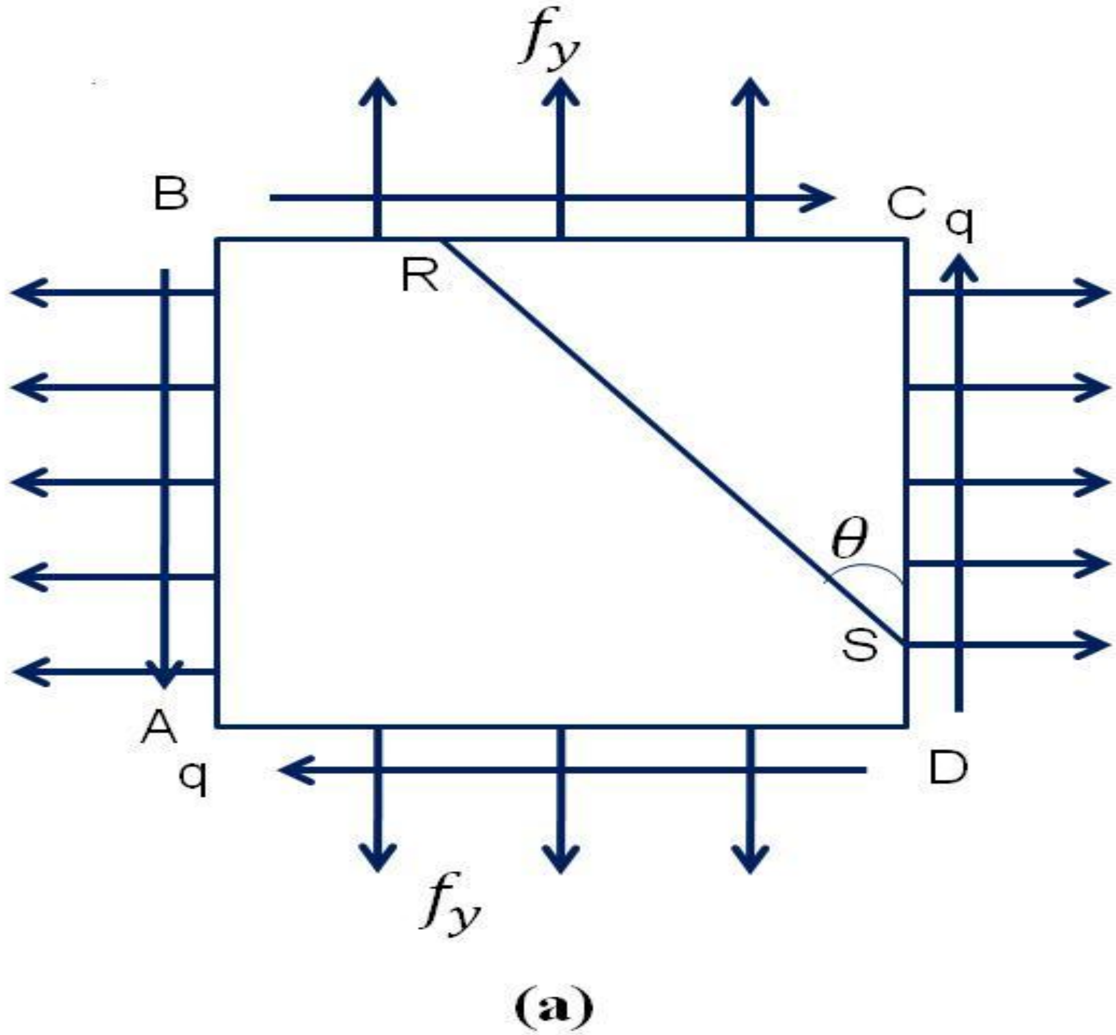
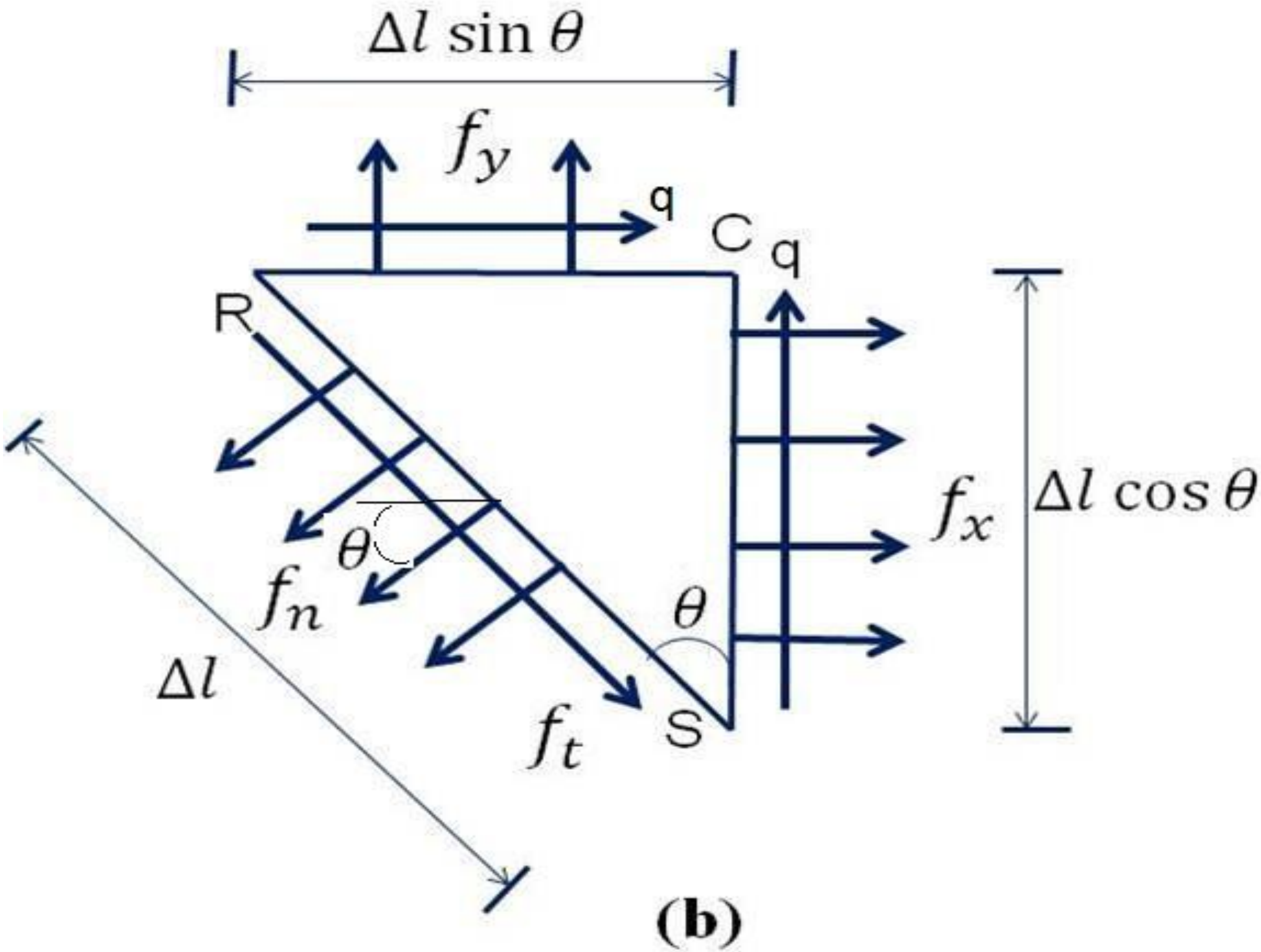


Fig.12.1



$\Delta l \sin\theta$
f_y
q
C
q
R
f_x
$\Delta l \cos\theta$
θ
f_n
θ
Δl
f_t
S
(b)

Fig.12.1

Let f_n and f_t be the normal and shear stress respectively acting on the plane RS as shown in Fig.12.1(b).

Now applying the horizontal equilibrium condition for the free body of portion RCS.

(i.e.,) sum of all the forces in X-direction is equal to zero $(\sum F_x = 0)$

$$f_x(\Delta l\ cos\theta \times 1) + q(\Delta l\ sin\theta \times 1)$$

$$-f_n\,(\Delta l \times 1)cos\theta + f_t\,(\Delta l \times 1)sin\theta = 0$$

i.e.,

$$f_x cos\theta + q sin\theta - f_n\, cos\theta + f_t\, sin\theta \;= 0 \quad ...(1)$$

similarly, applying the vertical equilibrium condition for the free body portion RCS.

i.e., sum of all the forces in Y-direction is equal to zero $\left(\sum F_y = 0\right)$

$$q(\Delta l\ cos\theta \times 1) + f_y(\Delta l\ sin\theta \times 1)$$

$$-f_n\,(\Delta l \times 1)sin\theta - f_t\,(\Delta l \times 1)cos\theta = 0$$

i.e.,

$$q cos\theta + f_y sin\theta - f_n \, sin\theta - f_t \, cos\theta = 0 \quad \ldots(2)$$

Rewriting equation (1) and (2)

We get,

$$f_n \, cos\theta - f_t \, sin\theta = f_x cos\theta + q \, sin\theta \qquad \ldots(3)$$

$$f_n \, sin\theta + f_t \, cos\theta = f_y \, sin\theta + q \, cos\theta \qquad \ldots(4)$$

Solving (3) and (4)

$(3) \times cos\theta + (4)\ sin\theta \ \Rightarrow$

$$f_n = f_x\, cos^2\theta + f_y\, sin^2\theta + 2q\ sin\theta cos\theta \qquad \ldots(5)$$

Also, $(4) \times cos\theta - (3)\ sin\theta \ \Rightarrow$

$$f_t = f_y sin\theta cos\theta - f_x sin\theta cos\theta + q\ cos^2\theta - q sin^2\theta$$

i.e., $f_t = \frac{f_y - f_x}{2} \sin 2\theta + q(cos^2\theta - sin^2\theta) \qquad \ldots(6)$

Eqn (5) $\Rightarrow$

$$f_n = f_x\left(\frac{1+\cos 2\theta}{2}\right) + f_y\left(\frac{1-\cos 2\theta}{2}\right) + q \sin 2\theta$$

$$\therefore f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2} \cos 2\theta + q \sin 2\theta \qquad \ldots(7)$$

And

$$f_t = \frac{f_y - f_x}{2} \sin 2\theta + q \cos 2\theta \qquad \ldots(8)$$

Eqn (7) and (8) gives the expression for normal and shear (tangential) stresses at a point on any plane inclined at an angle θ to the reference (vertical plane) plane

Sign convention

Orientation of the plane:

Anticlockwise direction is positive if measured from vertical plane

Normal stresses : Tensile positive

Shear stresses :

Shear stress on vertical plane which tends to rotate the element in anticlockwise direction is positive.

i.e., sign convention as shown in Fig.12.1 is positive

Example 12.1: At a point in a strained material the normal stresses on two planes at right angles are 80 N/mm^2 and 60 N/mm^2 both tensile. Find the resultant intensity of stress on a plane inclined at 60˚ to the axis of maximum stress.

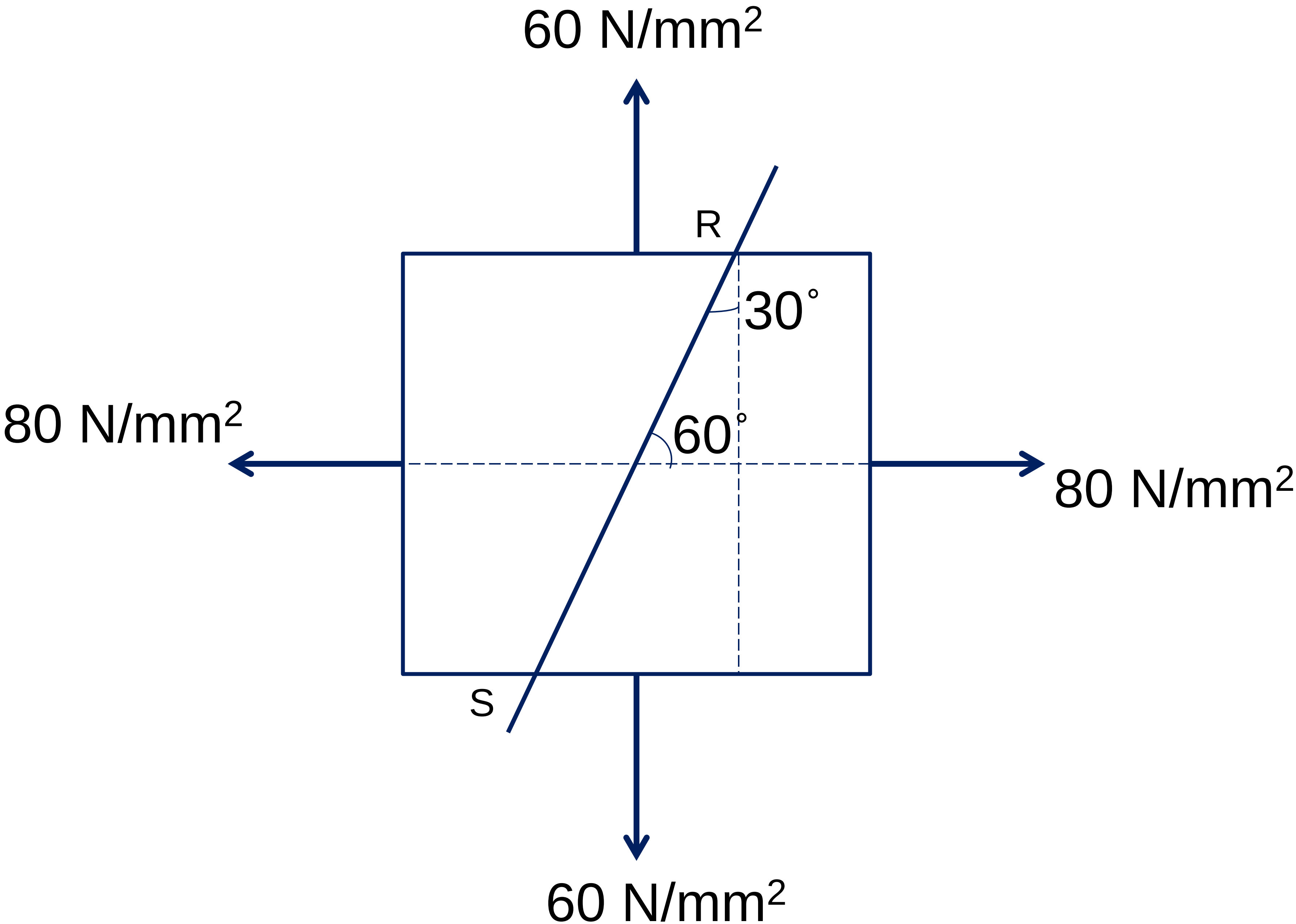
Soln:

Given, $f_x = 80\ N/mm^2$

$f_y = 60\ N/mm^2$

$q = 0$

$\theta = -30°$


60 N/mm2
R
30°
80 N/mm2
60°
80 N/mm2
S
60 N/mm2

Normal stress on the inclined plane,

$$f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2}\cos 2\theta + q\sin 2\theta$$

$$= \frac{80+60}{2} + \frac{80-60}{2}\cos(-60)$$

$$= 70 + 10\cos(60)$$

$$= 75\ N/mm^2$$

$$\therefore f_n = 75\ N/mm^2$$

Tangential (shear) stress on the inclined plane,

$$f_t = \frac{f_y - f_x}{2} \sin 2\theta + q \cos 2\theta$$

$$= \frac{60-80}{2} \sin(-60)$$

$$= 10 \sin(60)$$

$$= 8.66 \; N/mm^2$$

$$\therefore f_t = 8.66 \; N/mm^2$$

$\therefore$Resultant stress, $f_r = \sqrt{{f_n}^2 + {f_t}^2}$

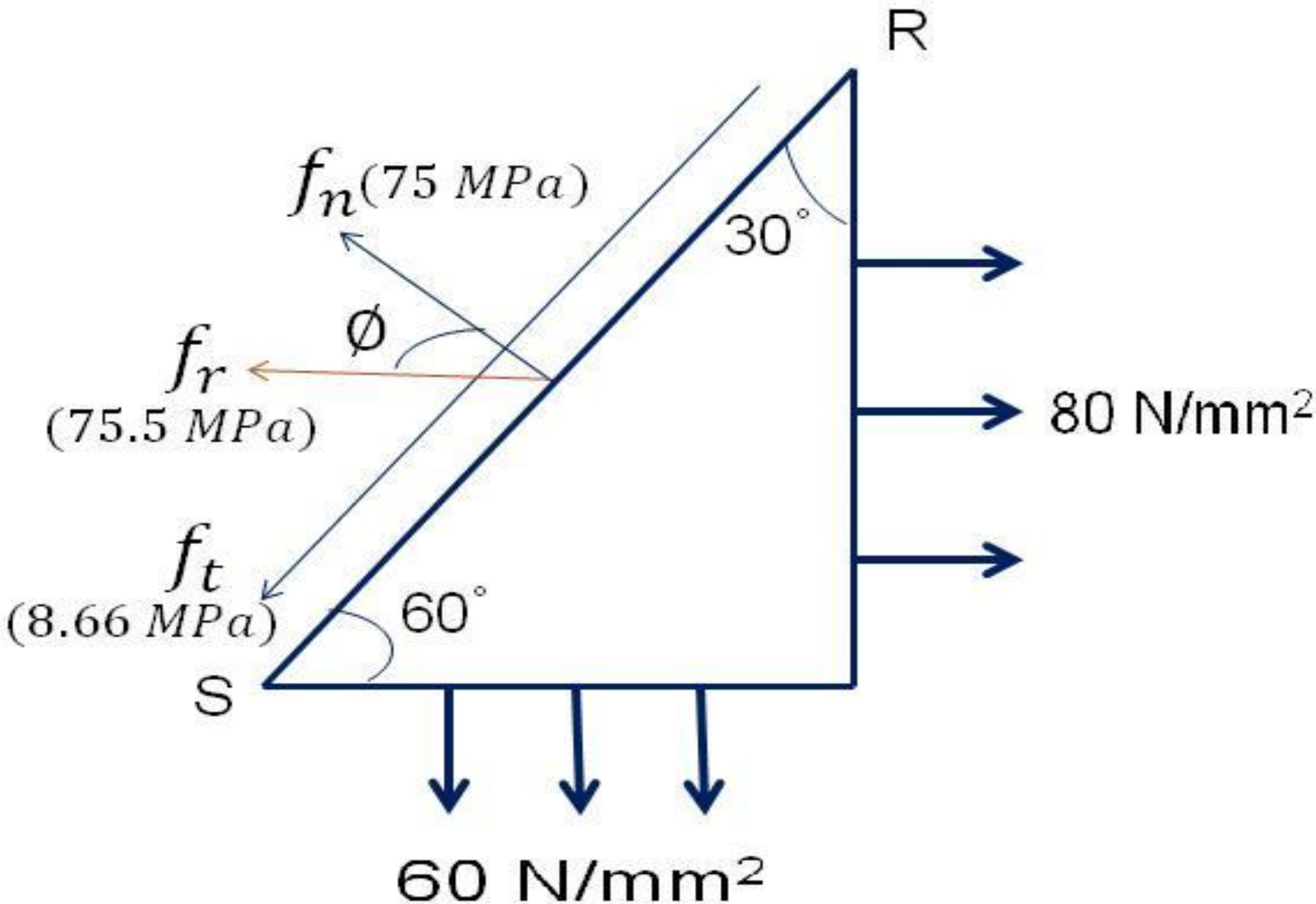
$$= \sqrt{75^2 + 8.66^2}$$

$$= 75.5\ N/mm^2$$

If ∅ is the angle made by the resultant stress with normal stress, then

$$\tan \emptyset = \frac{f_t}{f_n}$$

$$\therefore\ \emptyset = tan^{-1}\left(\frac{8.66}{75}\right) = 6°35'$$


R
$f_n (75\ MPa)$
30°
Ø
f_r
$(75.5\ MPa)$
80 N/mm²
f_t
$(8.66\ MPa)$
60°
S
60 N/mm²

- **Principal Stresses and principal planes**
- From transformation equations, it is clear that the normal and shear stresses vary continuously with the orientation of planes through the point.
- Among those varying stresses, finding the maximum and minimum values and the corresponding planes are important from the design considerations.

Out of all possible planes, on one plane (major principal plane) the normal stress is maximum, called major principal stress and on one plane (minor principal plane) the normal stress is minimum, called minor principal stress

The *principal stresses* occur on planes, and thus in directions, that are 90° from each other. The direction of the maximum or minimum normal stress is found by taking the derivative of f_n with respect to angle θ.

Expression for principal stresses and orientation of principal planes

We know,

$$f_n = \frac{f_x+f_y}{2} + \frac{f_x-f_y}{2}\cos 2\theta + q\sin 2\theta \text{ ------(7)}$$

In the above expression f_n is a function of θ.

Therefore for f_n to be maximum or minimum,

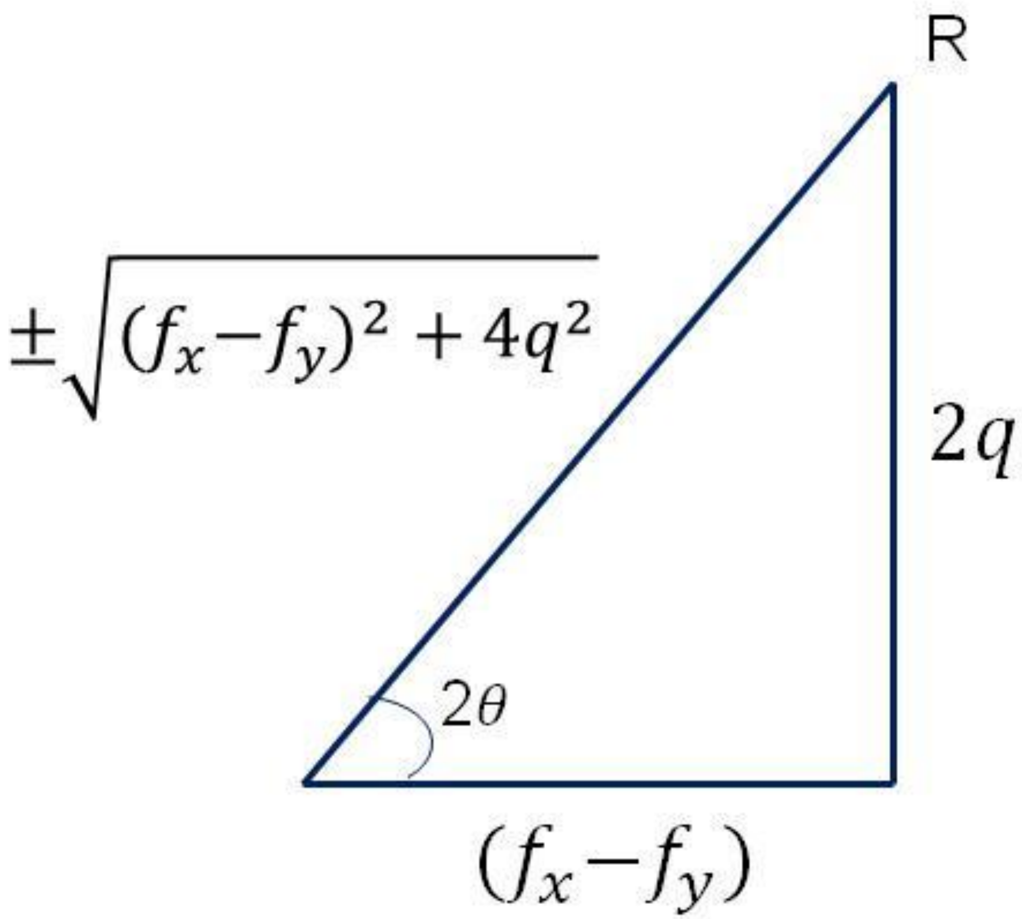
$$\frac{d(f_n)}{d\theta} = 0$$

Then differentiating equation (7) and equate it to zero.

i.e., $\frac{f_x - f_y}{2}(-\sin 2\theta)2 + 2q\cos 2\theta = 0$

$$q\cos 2\theta = \frac{f_x - f_y}{2}(\sin 2\theta)$$

$$\tan 2\theta = \frac{2q}{f_x - f_y} \qquad \ldots(9)$$


R
$\pm\sqrt{(f_x - f_y)^2 + 4q^2}$
$2q$
2θ
$(f_x - f_y)$

As reference to equation (9), we have to develop the expression for $\cos 2\theta$ and $\sin 2\theta$ as follows:

$$\cos 2\theta = \pm \frac{f_x - f_y}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

And $$\sin 2\theta = \pm \frac{2q}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

Substitute the value of $\sin 2\theta$ and $\cos 2\theta$ in equation (7) for f_n to get the maximum and minimum normal stresses (called major and minor principal stresses).

$$\therefore f_{1,2} = \left(\frac{f_x + f_y}{2}\right) \pm \left(\frac{f_x - f_y}{2}\right) \frac{(f_x - f_y)}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

$$\pm\, q \frac{2q}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

$$f_{1,2} = \left(\frac{f_x + f_y}{2}\right) \pm \frac{\frac{(f_x - f_y)^2}{2} + 2q^2}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

$$= \left(\frac{f_x + f_y}{2}\right) \pm \frac{(f_x - f_y)^2 + 4q^2}{2\sqrt{(f_x - f_y)^2 + 4q^2}}$$

$$= \left(\frac{f_x + f_y}{2}\right) \pm \frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2}$$

i.e., major principal stress,

$$f_1 = \frac{f_x + f_y}{2} + \frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2} \quad \text{....(10)}$$

And minor principal stress,

$$f_2 = \frac{f_x + f_y}{2} - \frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2} \quad \text{....(11)}$$

The orientation of the planes α is given by,

$$\tan 2\alpha = \frac{2q}{f_x - f_y}$$

$$\tan(2\alpha + \pi) = \tan 2\alpha = \frac{2q}{f_x - f_y}$$

$\therefore$ The orientation of the plane is α (or) $\left[\alpha + \frac{\pi}{2}\right]$

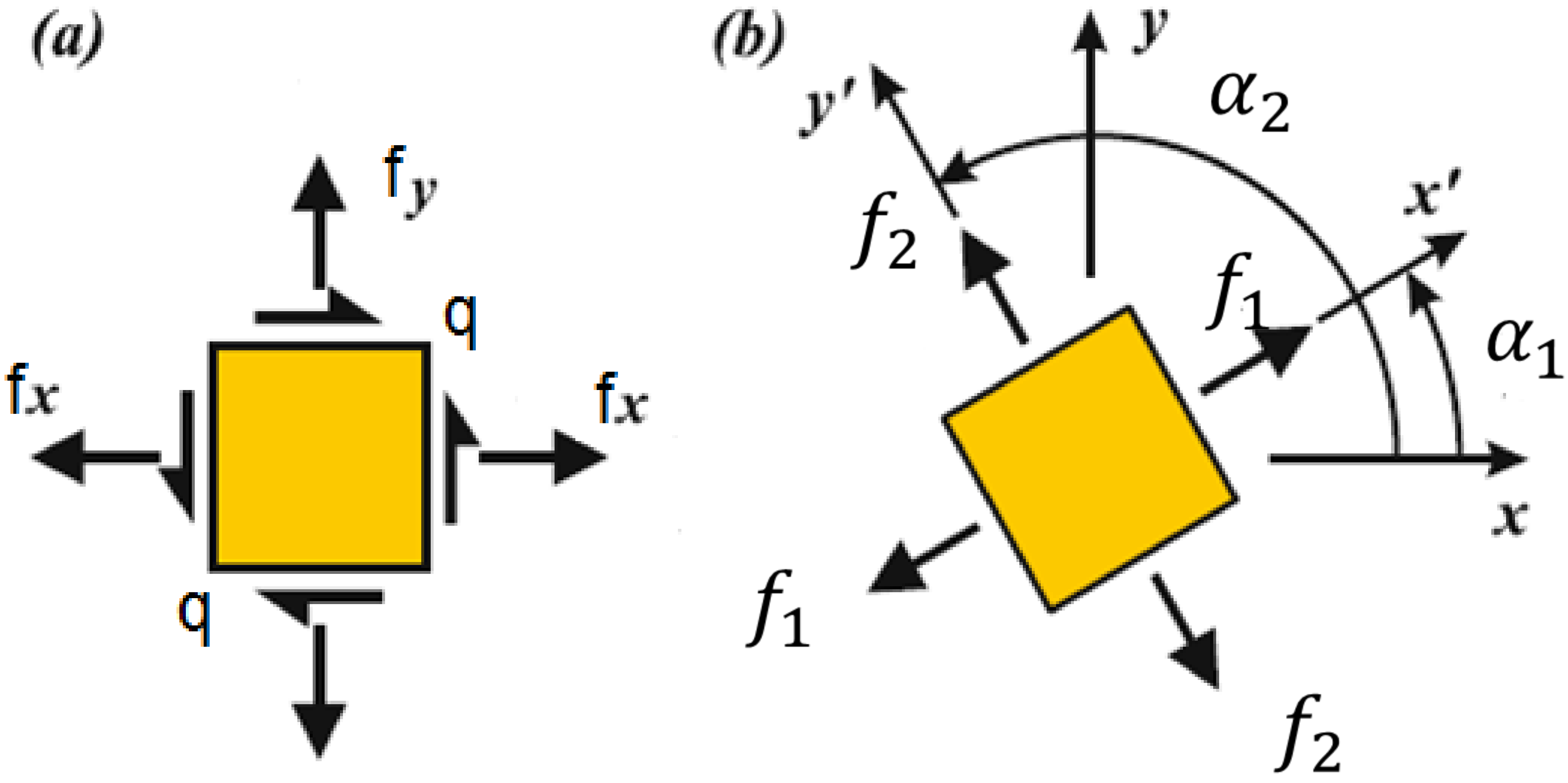


Figure 12.4 **(a)** The original stress state. **(b)** The principal stresses and their directions.

On the principal planes, the shear stress is

$$f_{t\,1,2} = \frac{f_y - f_x}{2}\left\{\pm\frac{2q}{\sqrt{(f_x-f_y)^2+4q^2}}\right\} + q\left\{\pm\frac{f_x - f_y}{\sqrt{(f_x-f_y)^2+4q^2}}\right\} = 0$$

for both the values

∴ The principal planes are the planes on which the shear stress is zero and normal stress in maximum or minimum.

Maximum Shear Stress

It is also useful to know the ***maximum in-plane shear stress*** that an element experiences. The shear stress transformation equation f_t is differentiated with respect to θ and the result set equal to zero to find the angle at which the *in-plane shear stress* is a maximum

we know,

$$f_t = \frac{f_y - f_x}{2} \sin 2\theta + q \cos 2\theta$$

For the plane on which shear stress maximum or minimum,

$$\frac{d(f_t)}{d\theta} = 0$$

i.e., $\frac{f_y - f_x}{2} 2\cos 2\theta - 2q \sin 2\theta = 0$

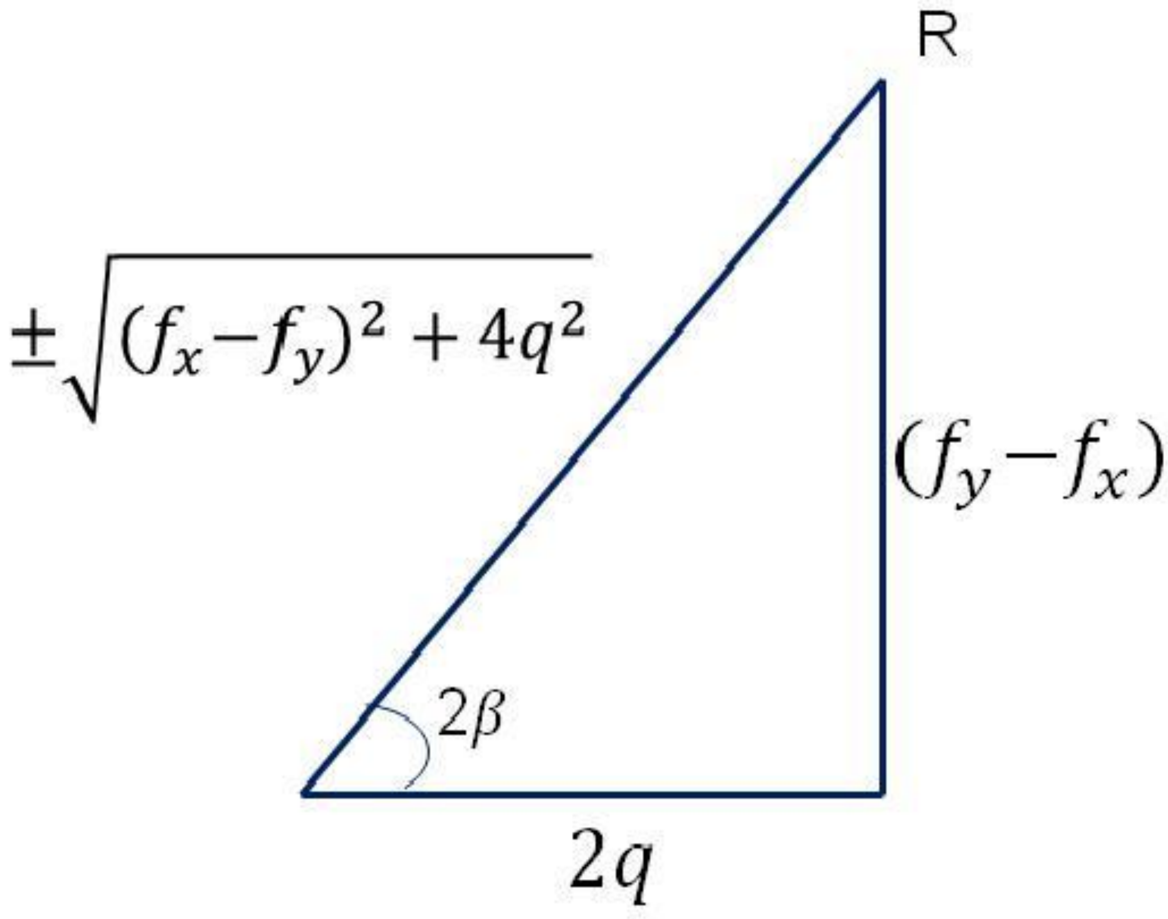
i.e., $\tan 2\theta = \frac{f_y - f_x}{2q}$

Referring θ as β

$$\tan 2\beta = \frac{f_y - f_x}{2q} \quad \ldots.(12)$$

$$\tan 2\beta = \tan(\pi + 2\beta)$$

The orientation is β (or) $\left[\beta + \frac{\pi}{2}\right]$


R
$\pm\sqrt{(f_x - f_y)^2 + 4q^2}$
$(f_y - f_x)$
2β
$2q$

Referring to this we have to find

$sin\, 2\beta \;\; and \cos 2\beta$

i.e., $$sin\, 2\beta \;=\; \pm\frac{f_y - f_x}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

$$\cos 2\beta \;=\; \pm\frac{2q}{\sqrt{(f_x - f_y)^2 + 4q^2}}$$

$$\therefore\ f_t = \frac{f_y - f_x}{2}\left\{\pm\frac{f_y - f_x}{\sqrt{(f_x - f_y)^2 + 4q^2}}\right\}$$

$$+q\left\{\pm\frac{2q}{\sqrt{(f_x - f_y)^2 + 4q^2}}\right\}$$

$$= \pm\frac{(f_x - f_y)^2 + 4q^2}{2\sqrt{(f_x - f_y)^2 + 4q^2}}$$

$$\therefore (f_t)\,\max or\ min = \pm\frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2} \quad .(14)$$

$$(f_t)\,\max = \frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2}$$

and $(f_t)min = -\frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2}$

Relationship between principal planes and maximum shear planes

We know, $\tan\left(2\alpha + \frac{\pi}{2}\right) = -\cot 2\alpha \quad = -(\frac{f_x - f_y}{2q})$

$$= \frac{f_y - f_x}{2q} \qquad = \tan 2\beta$$

$$\therefore 2\alpha + \frac{\pi}{2} = 2\beta$$

$$\alpha + \frac{\pi}{4} = \beta \quad ....(13)$$

i.e.., The maximum shear planes are inclined at 45° to the principal planes.

Note:

1) The normal stress on the planes on which the shear stress is maximum or minimum

We know, $f_n = \frac{f_x+f_y}{2} + \frac{f_x-f_y}{2}\cos 2\theta + q\sin 2\theta$

$$f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2}\left\{\pm\frac{2q}{\sqrt{(f_x-f_y)^2 + 4q^2}}\right\} + q\left\{\pm\frac{f_y - f_x}{\sqrt{(f_x-f_y)^2 + 4q^2}}\right\}$$

$$= \frac{f_x+f_y}{2}$$

2) From equations (10) and (11)

$f_1 + f_2 = f_x + f_y$ (stress invariants)

3) From equations (10), (11) and (14)

$$\frac{f_1 - f_2}{2} = (f_t)_{max}$$

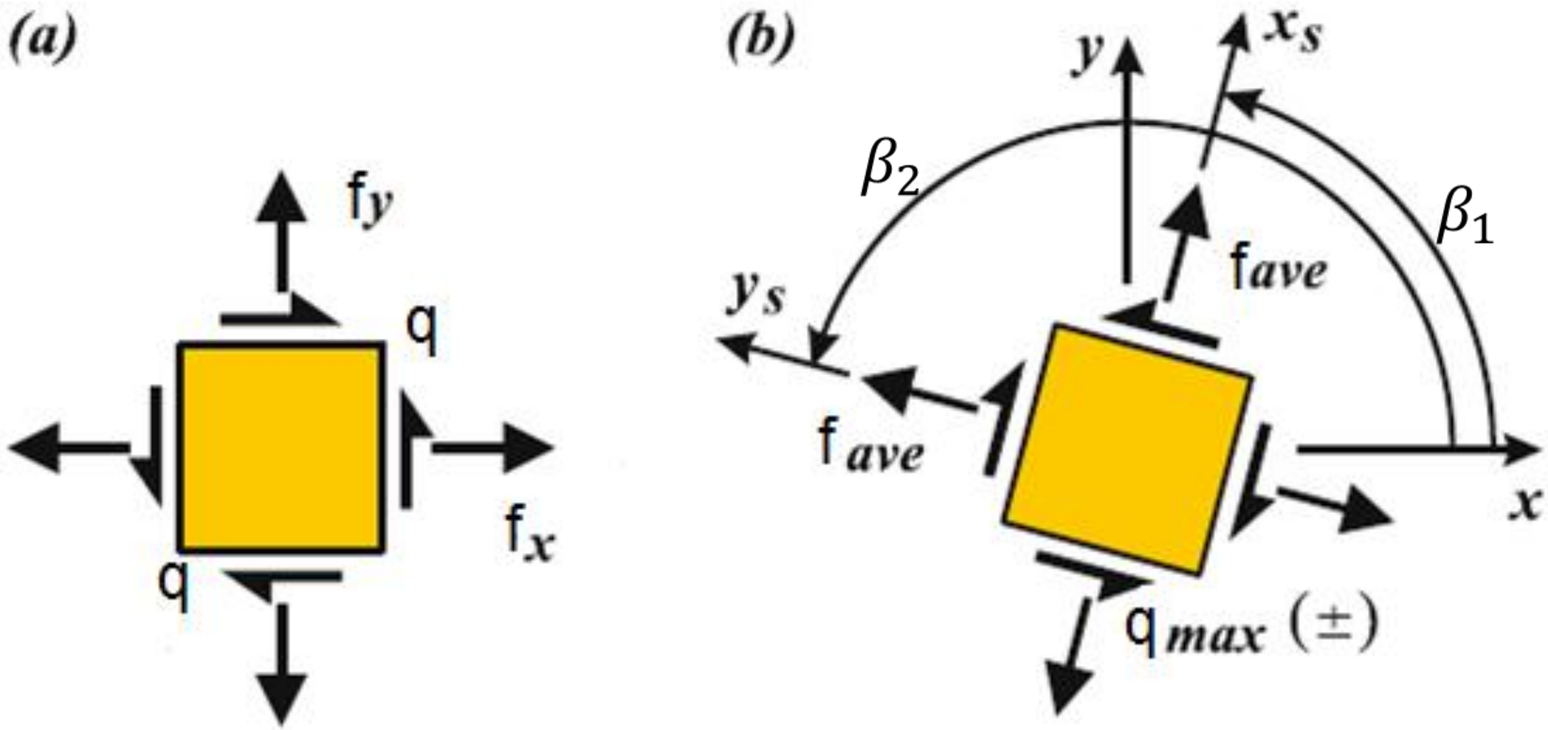


Figure 12.6 **(a)** The original stress state.
(b) The maximum shear stresses and the angle of the planes in which they act.

Special Cases

1)

When the material is subjected to only f_x

Then the normal stress at a point on any plane,

$$f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2}\cos 2\theta + q\sin 2\theta$$ becomes

$$f_n = \frac{f_x}{2} + \frac{f_x}{2}\cos 2\theta$$

$$= \frac{f_x}{2}(1 + \cos 2\theta)$$

$$= f_x\, cos^2\theta$$

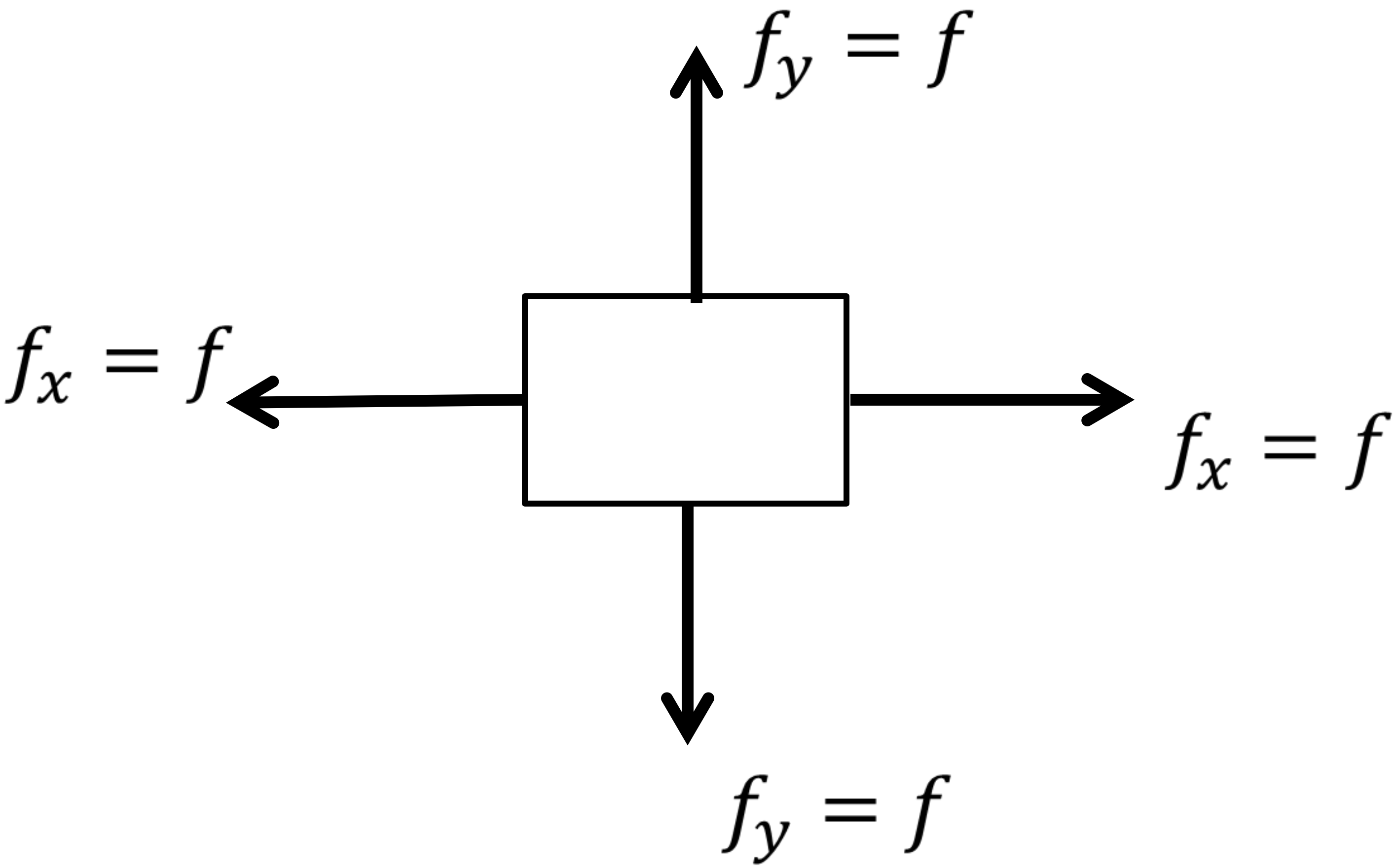
$f_1 = f_x$ and $f_2 = 0$

i.e., in this case major principal stress $f_1 = f_x$

minor principal stress $f_2 = 0$ and

maximum shear stress $(f_t)_{max} = \frac{f_x}{2}$

2) When $f_x = f_y = f$



$$f_x = f_y = f$$

$$\text{and } q = 0$$

$$\therefore f_n = f \text{ and } f_t = 0$$

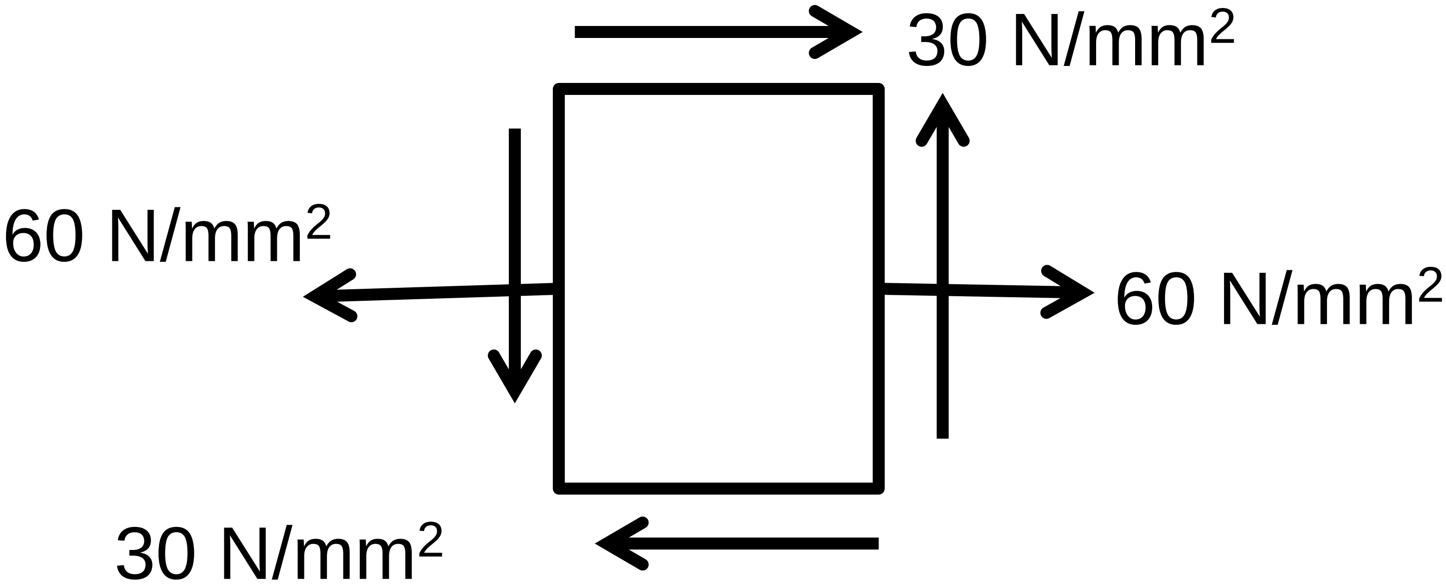
This state of stress is called isotropic state of stress.

Isotropic point is the point at which the normal stress is same and shear stress is zero in all the directions

Ex:110. At a point in the web of a girder the bending stress is 60 N/mm^2 tensile and the shearing stress at the same point is 30 N/mm^2 . Determine

a) Principal stresses and principal planes

b) Maximum shear stress and its orientations



Solution:

Given, $f_x = 60$ N/mm^2

$f_y = 0$

$q = 30$ N/mm^2

a) Principal stresses:

$$f_{1,2} = \frac{f_x+f_y}{2} \pm \frac{1}{2}\sqrt{(f_x-f_y)^2 + 4q^2}$$

$$f_{1,2} = \frac{60+0}{2} \pm \frac{1}{2}\sqrt{60-0)^2 + 4(30)^2}$$

$$= 30 \pm 42.43$$

$f_1 = 72.43\ N/mm^2$ (Major principal stress)

$f_2 = -12.43\ N/mm^2$ (Minor principal stress)

Principal planes

$$\tan 2\alpha = \frac{2q}{f_x - f_y}$$

$$= \frac{2\times30}{60}$$

$$= 1$$

$$\therefore 2\alpha = 45°$$

i.e., $\alpha = 22.5°$

To find the normal stress on the plane at 22.5°, put $\alpha = 22.5°$ in the equation for normal stress.

We get,

$f_n\ (at\ \theta = 22.5°) = 30$ +30cos(45) +

30sin(45)

= 72.43 N/mm^2

= major principal stress

$\therefore \alpha_1 = 22.5°\ \ and\ \ \alpha_2 = 22.5 \pm 90$

$= 112.5°\ or\ -67.5°$

(b) **Maximum shear stress,**

$$q_{max} = \frac{f_1 - f_2}{2}$$

$$q_{max} = \frac{72.43 - (-12.43)}{2} = 42.43\ N/mm^2$$

Orientation of maximum shear plane

$$\tan 2\beta \ = \frac{f_y - f_x}{2q} = \frac{0 - 60}{2 \times 30} = -1$$

$$\therefore \beta = -22.5°$$

To find the nature of maximum shear stress, put

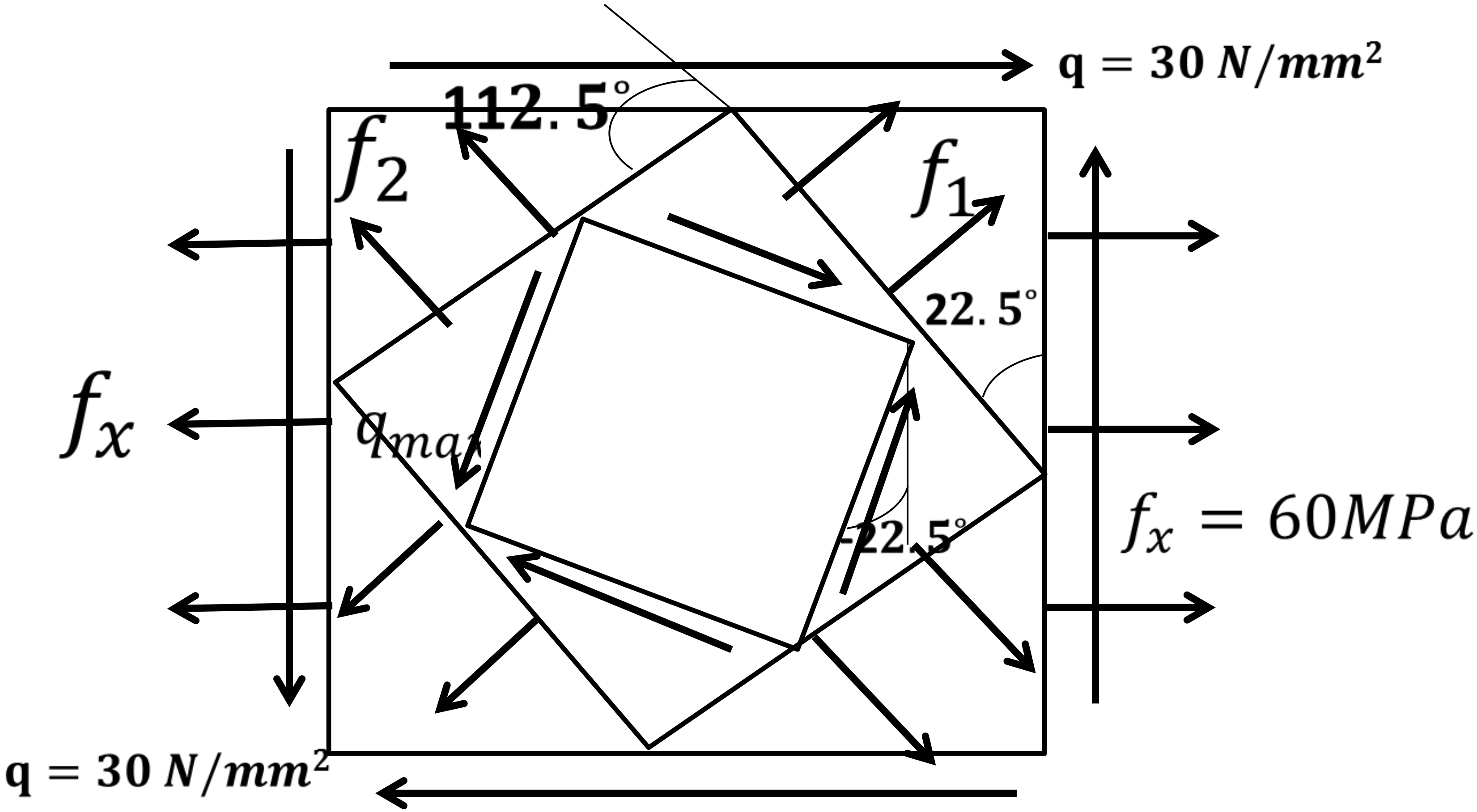
$\theta = -22.5° \, in\ the\ equation\ for\ shear\ stress,$

$we\ get$

$f_t(at\ \theta = -22.5°)$

$$= \frac{0 - 60}{2}\sin(-45) + 30\cos(-45)$$

= 42.43N/mm^2



$\mathbf{q} = \mathbf{30}\ N/mm^2$
$112.5°$
f_2
f_1
$22.5°$
f_x
q_{max}
$-22.5°$
$f_x = 60MPa$
$\mathbf{q} = \mathbf{30}\ N/mm^2$

<u>Ex:111.</u> At a point in a strained material the resultant intensity of stress across a plane is 113.14 N/mm^2 tensile inclined at 45° to its normal as shown in Fig. 12.9. The normal component of intensity across the plane at right angles is 30 N/mm^2 compressive. Find the position of principal planes and stresses across them. Find also the value of maximum shear at that point.

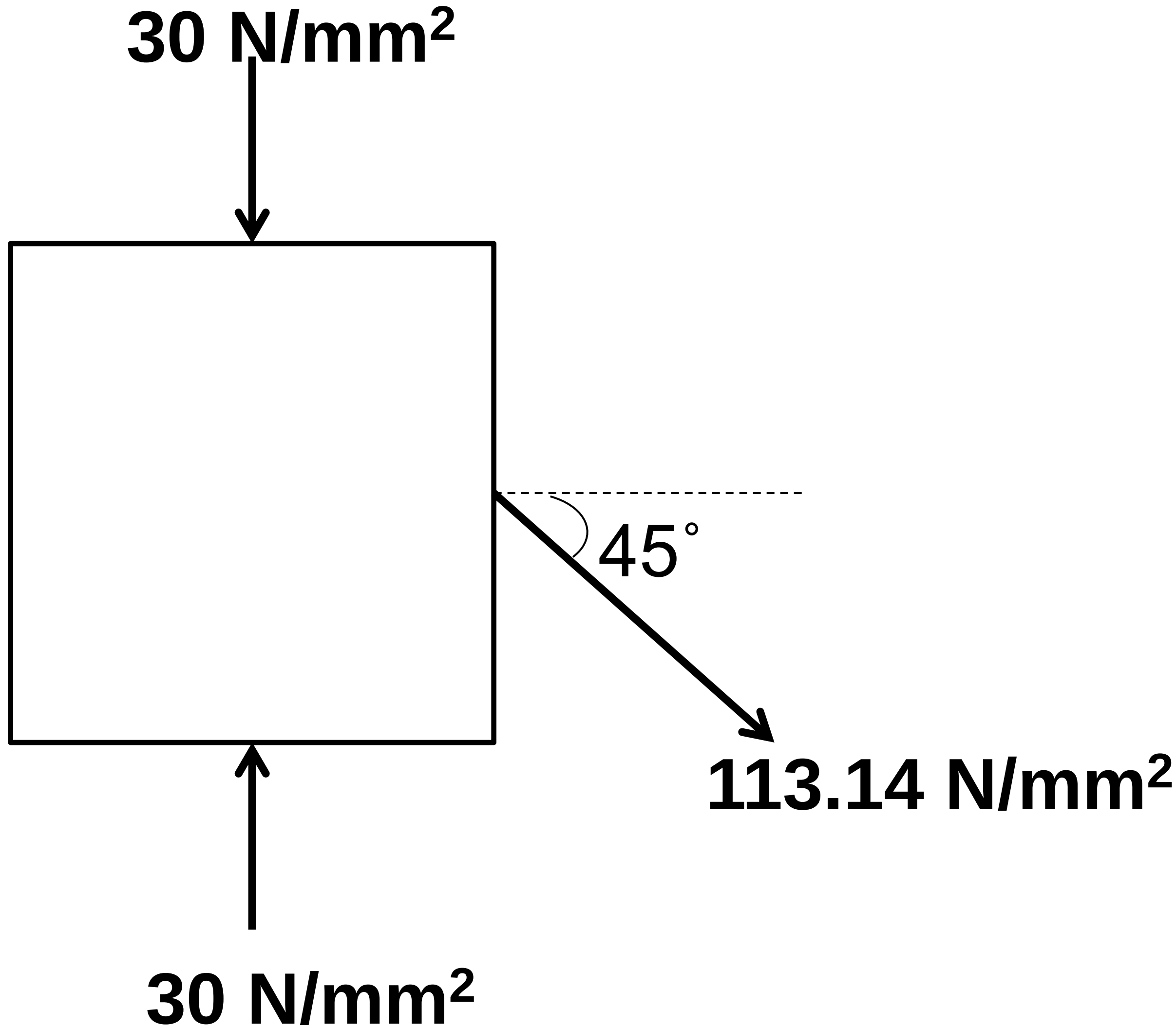


Fig.12.9

Soln:

$$f_x = 113.14 \ \cos 45$$

$$= 80 \ N/mm^2$$

$$q = 113.14 \ \sin 45$$

$$= 80 \ N/mm^2$$

Then the equivalent stress system is as shown in Fig. 12.10

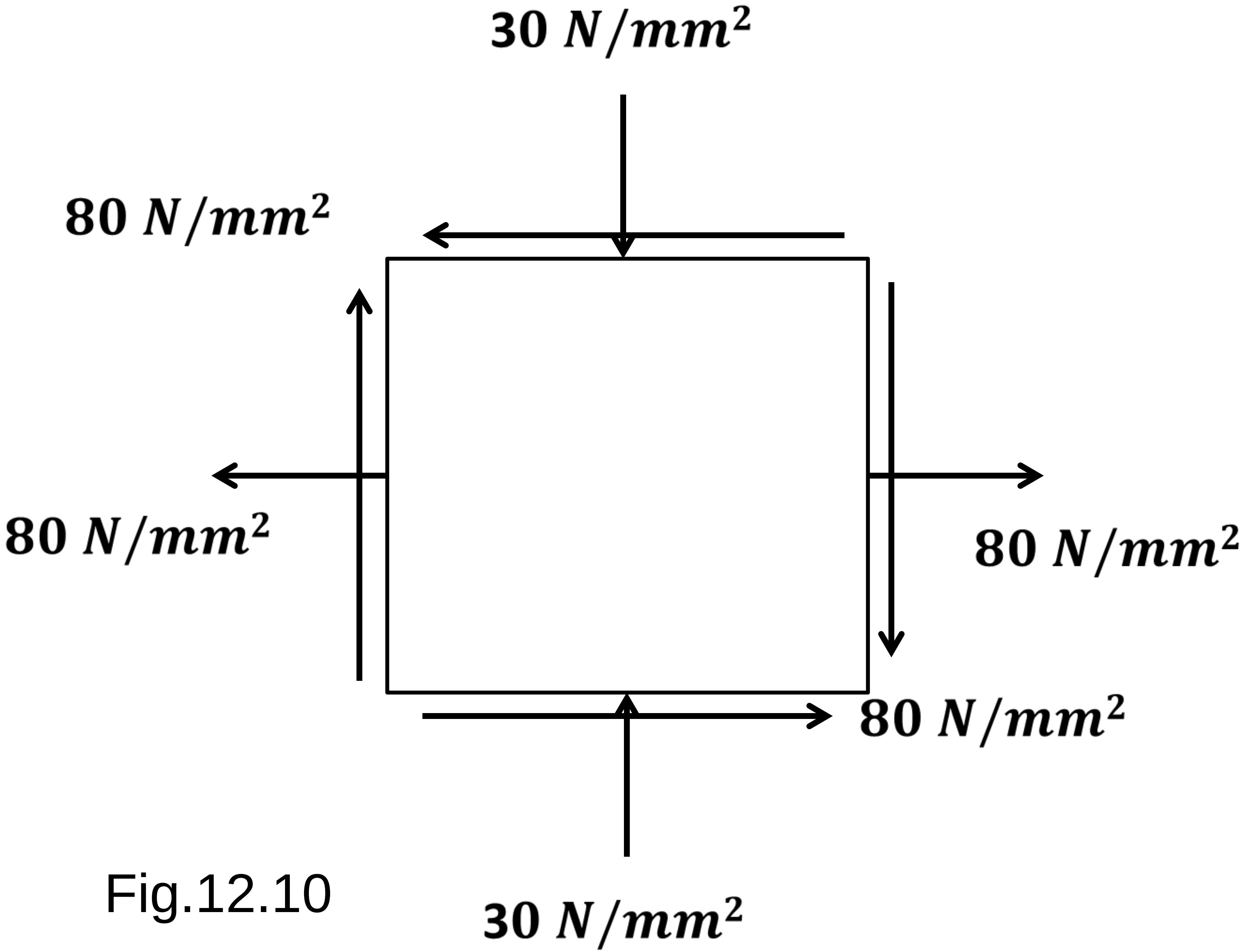


Fig.12.10

$$f_x = 80\ N/mm^2$$

$$f_y = -30\ N/mm^2$$

$$q = -80\ N/mm^2$$

Principal stress

$$f_{1,2} = \frac{f_x + f_y}{2} \pm \frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2}$$

$$= \frac{80 - 30}{2} \pm \frac{1}{2}\sqrt{(80 + 30)^2 + 4(-80)^2}$$

$$f_{1,2} = 25 \pm 97.08$$

$$\therefore f_1 = 122.08\ N/mm^2$$

$$f_2 = -72.08\ N/mm^2$$

$\therefore$ Major principal stress = 122.08 N/mm^2 (tensile)

Minor principal stress = 72.08 N/mm^2 (compressive)

Principal plane

$$\tan 2\alpha = \frac{2q}{f_x - f_y} = \frac{2(-80)}{80-(-30)} = -1.4545$$

$$2\alpha = -55.49^\circ$$

$$\alpha = -27.75^\circ \;\; or \;\; -27.75^\circ + 90$$

$$= 62.25^\circ$$

$$\therefore \alpha = -27.75^\circ \;\; or \;\; 62.25^\circ$$

Orientation of principal plane

Put $\alpha = 62.25^\circ$ in general equation of normal stress

$$f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2}\cos 2\theta + q\sin 2\theta$$

$$f_n = \frac{80 - 30}{2} + \frac{(80 + 30)}{2} \cos\ (2 \times 62.25^\circ)$$

$$+ (-80) \sin(2 \times 62.25^\circ)$$

$$= -72.08\ N/mm^2$$

Therefore the minor principal stress is at an angle of $62^\circ 15'$

Maximum shear stress,

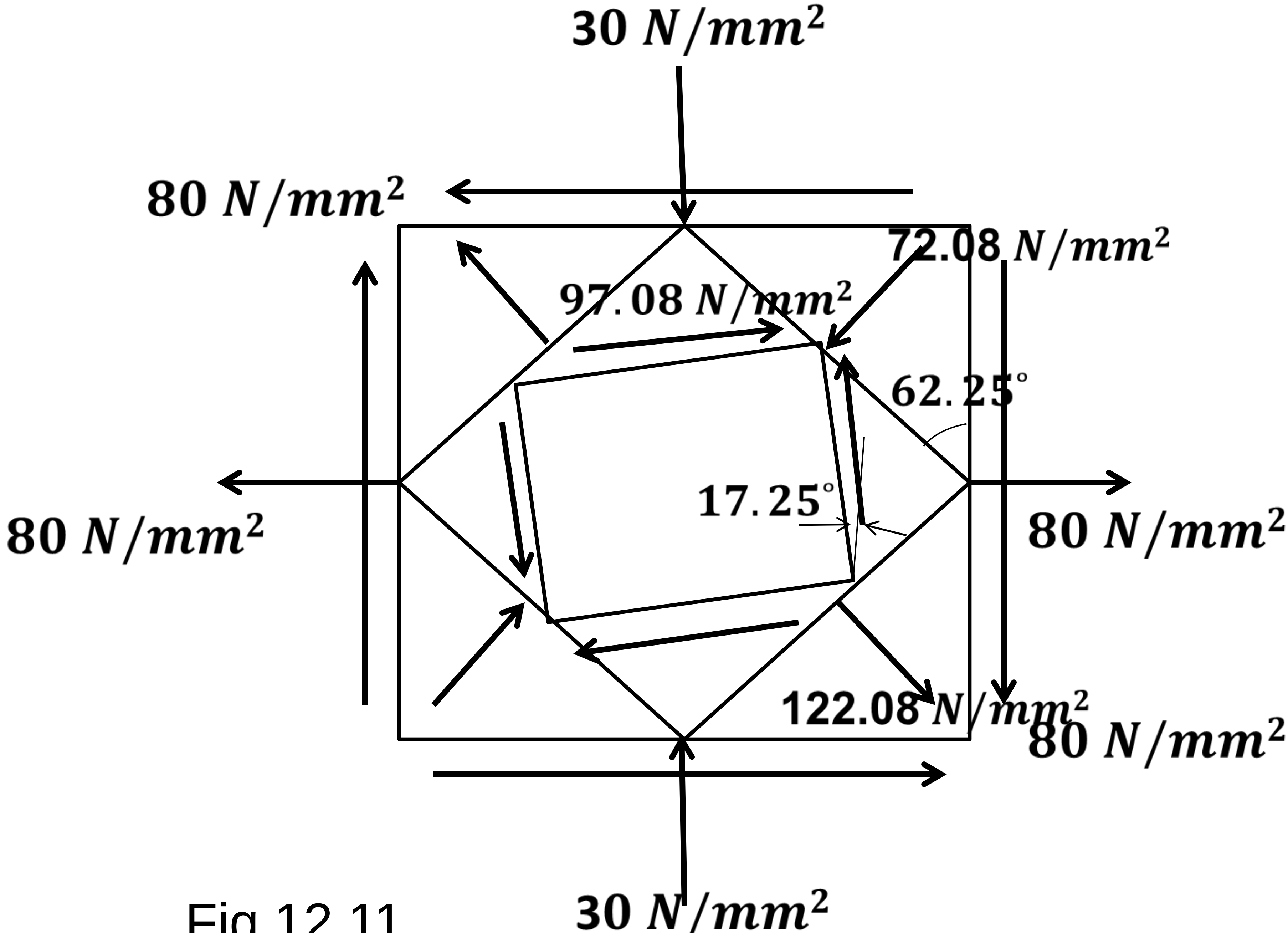
$$q_{max} = \frac{f_1 - f_2}{2} = \frac{122.08-(-72.08)}{2} = 97.08\ N/mm^2$$

Orientation,

$$\tan 2\beta \ = \frac{f_y - f_x}{2q} = \frac{-30 - 80}{-2(80)}$$

$$2\beta = 34.50^{\circ}$$

$$\therefore \beta = 17.25^{\circ}$$



30 N/mm²
80 N/mm²
72.08 N/mm²
97.08 N/mm²
62.25°
17.25°
80 N/mm²
80 N/mm²
122.08 N/mm²
80 N/mm²
30 N/mm²

Fig.12.11

GRAPHICAL METHOD:

MOHR'S CIRCLE OF STRESSES

We know,

$$f_n = \frac{f_x+f_y}{2} + \frac{f_x-f_y}{2}\cos 2\theta + q\sin 2\theta$$

$$i.e.,\ f_n - \frac{f_x+f_y}{2} = \frac{f_x-f_y}{2}\cos 2\theta + q\sin 2\theta \quad \ldots.(1)$$

and $f_t = \dfrac{f_y-f_x}{2}\sin 2\theta + q\cos 2\theta \quad \ldots.(2)$

Squaring both sides of equations (1) and (2) and adding, we get

$$\left\{f_n - \frac{f_x + f_y}{2}\right\}^2 + {f_t}^2$$

$$= \left(\frac{f_x - f_y}{2}\right)^2 \cos^2 2\theta + q^2 \sin^2 2\theta$$

$$+2\,\frac{(f_x - f_y)}{2} \cos 2\theta\,.\,q \sin 2\theta + \left(\frac{f_x - f_y}{2}\right)^2 \sin^2 2\theta$$

$$+ q^2 \cos^2 2\theta - 2\,\frac{(f_x - f_y)}{2} \sin 2\theta\,.\,q \cos 2\theta$$

$$= \left(\frac{f_x - f_y}{2}\right)^2 + q^2$$

This is similar to equation of a circle

i.e., $(x-a)^2+(y-b)^2=r^2$

Comparing with this equation,

$$r=\sqrt{\left(\frac{f_x-f_y}{2}\right)^2+q^2}$$

$$=\frac{1}{2}\sqrt{(f_x-f_y)^2+4q^2}$$

The centre of the circle is $\left\{\frac{f_x+f_y}{2},0\right\}$

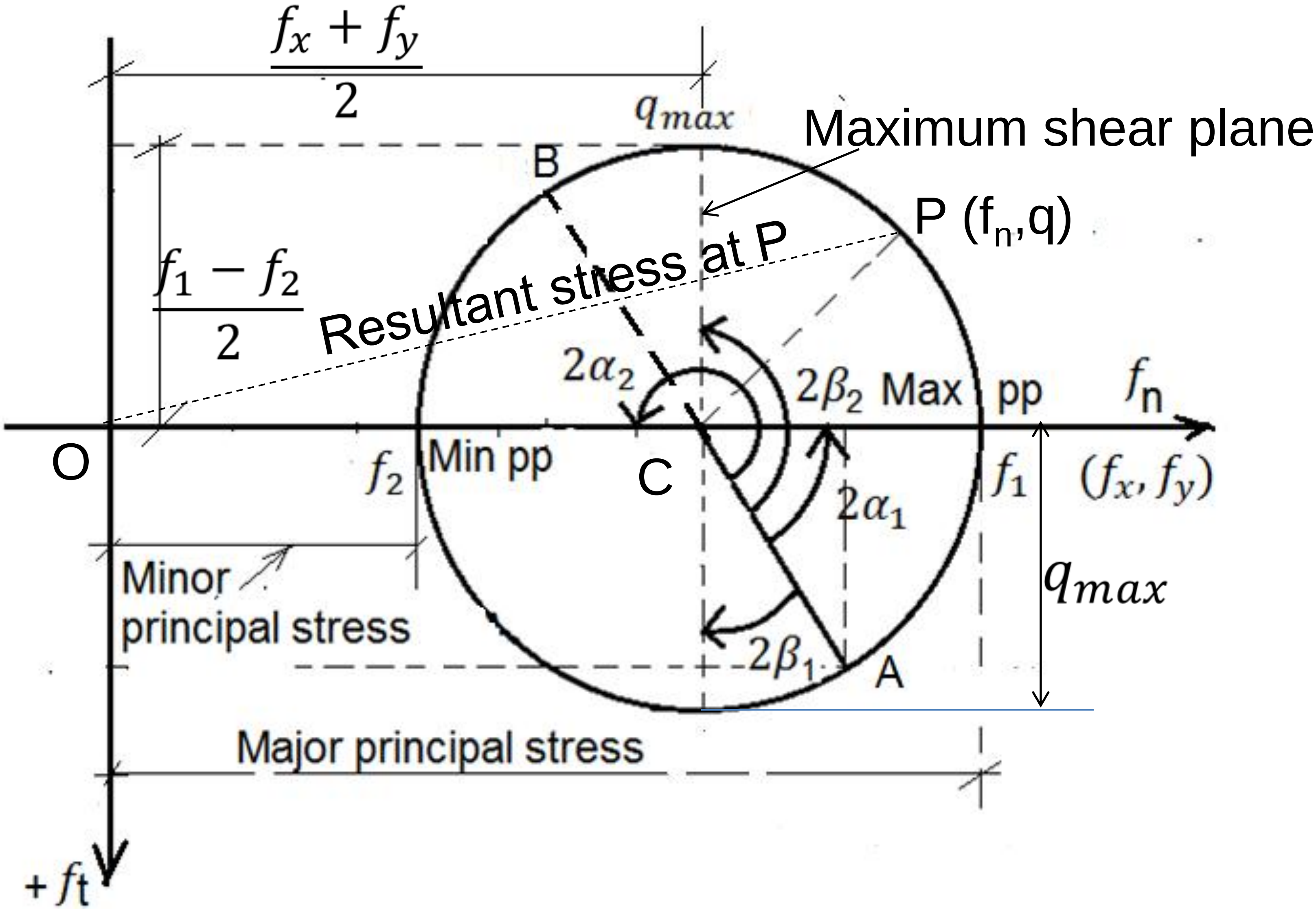

$\frac{f_x + f_y}{2}$
q_{max}
Maximum shear plane
B
P (f_n,q)
$\frac{f_1 - f_2}{2}$
Resultant stress at P
$2\alpha_2$
$2\beta_2$
Max pp
f_n
O
f_2
Min pp
C
f_1
(f_x, f_y)
$2\alpha_1$
Minor principal stress
q_{max}
$2\beta_1$
A
Major principal stress
+f_t

Fig.12.12

CONSTRUCTION OF MOHR'S CIRCLE

Sign convention

Normal stress:

Tensile positive

Shear stress:

Shear stresses on two opposite and parallel planes which tend to rotate the element in anticlockwise direction is positive.

Orientation of planes:

Anticlockwise with respect to the reference plane (corresponding to the state of stress on the vertical plane)

Construction of Mohr's circle…

Step 1: Select suitable (same) scale for normal stresses (X- axis) and shear stresses(Y- axis)

Step 2: Mark the points A and B on the graph corresponding to the state of stress on the vertical plane (A) and horizontal plane (B) by considering appropriate sign convention.

Step 3: Join A and B (which is the diameter of Mohr's circle) by a straight line, which cuts the normal stress axis (X axis) at C, which is the centre of circle.

Construction of Mohr's circle…

Step 4: With C as centre and AB as diameter draw a circle, which is the required circle representing the state of stress on the element.

Note:

- The circle cuts the normal stress axis at two points. The coordinates of these points are the principal stresses (maximum coordinate value is the major principal stress and minimum coordinate value is the minor principal stress) and the line connecting these points with centre of circle is the corresponding representation of principal planes.

Construction of Mohr's circle…

Note:

- The orientation of these planes can be measured with respect to the reference line CA (positive if anticlockwise)
- The angles measured from the graph are twice the actual angles.
- The vertical line passing through the centre of circle indicates the maximum shear plane and therefore maximum shear stress is equal to radius of the circle.
- To find the state of stress on any plane (P) at an angle θ from the vertical plane, mark the corresponding point (P) in the Mohr's circle by drawing a line at an angle 2θ from the reference line CA.

<u>Ex:112</u> An element in a strained body is as shown in Fig. 12.13

(i) Find the major and minor principal stresses and its corresponding principal planes.

(ii) Find the maximum shear stress and its corresponding planes.

(iii) Also find the normal and tangential stresses on an inclined plane at 60˚ to the vertical plane.

Solve by analytical method, and compare with graphical method.

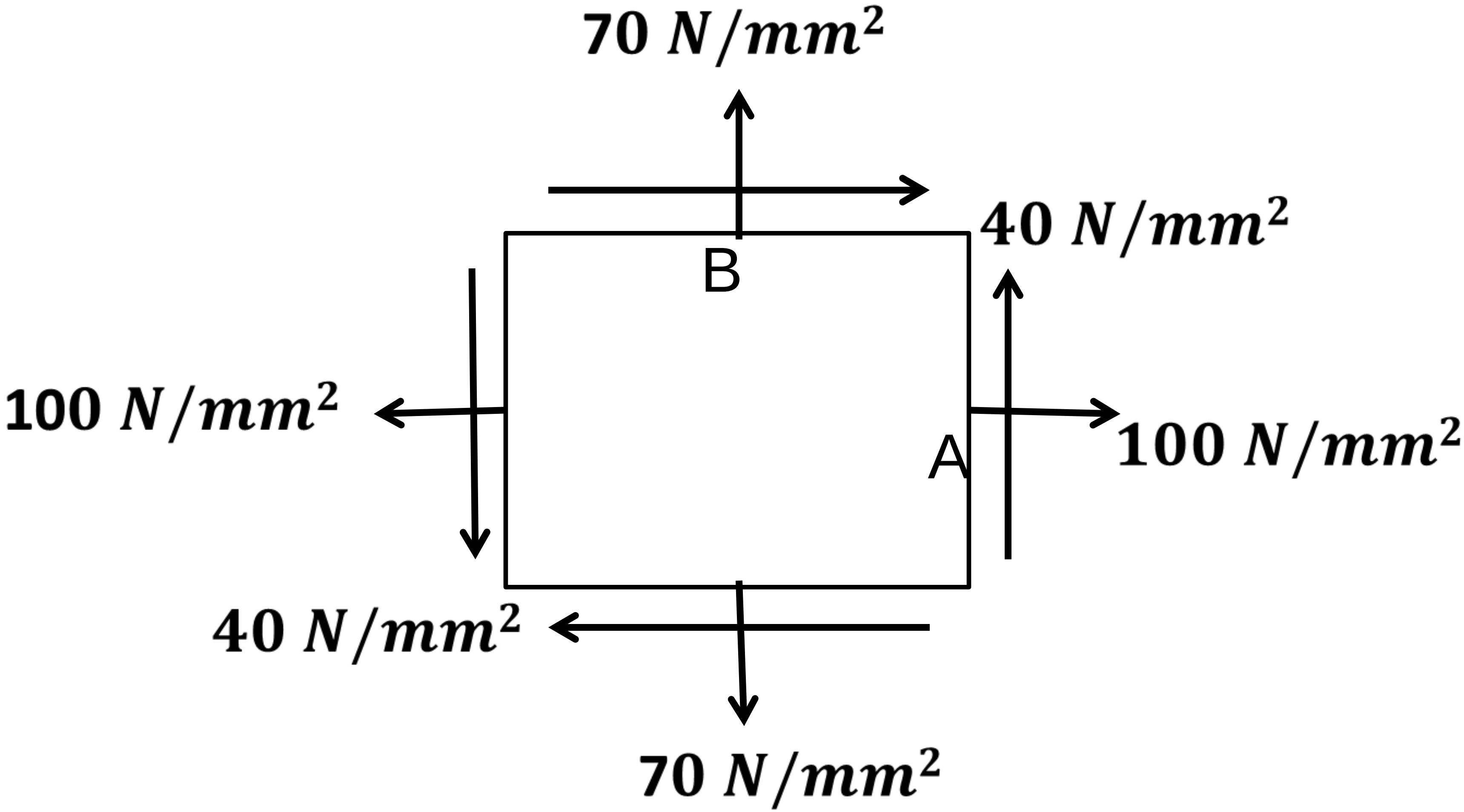


Fig.12.13

Soln:

ANALYTICAL METHOD

Given, f_x = 100 N/mm^2

f_y= 70 N/mm^2

q = 40 N/mm^2

(i) Principal stresses

(a)

$$f_{1,2} = \frac{f_x + f_y}{2} \pm \frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2}$$

$$f_{1,2} = \frac{100 + 70}{2} \pm \frac{1}{2}\sqrt{(100 - 70)^2 + 4(40)^2}$$

$$= 85 \pm 42.72$$

$f_1 = 127.72\ N/mm^2$ (Major principal stress)

$f_2 = 42.28\ N/mm^2$ (Minor principal stress)

Check: $f_x + f_y = f_1 + f_2$

i.e., 100 + 70 = 127.27 + 42.28

170 = 170

(b) Principal planes

We know, $\tan 2\alpha = \frac{2q}{f_x - f_y} = \frac{2(40)}{100-70} = 2.67$

$$\therefore \alpha = 34.72^\circ$$

Put $\alpha = 34.72^\circ$ in general equation of normal stress

$$f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2}\cos 2\theta + q\sin 2\theta$$

$$f_n = \frac{100+70}{2} + \frac{100-70}{2}\cos\ (2 \times 34.72)$$

$$+ (40)\sin(2 \times 34.72)$$

$$= 127.72\ N/mm^2$$

$$= f_1$$

$$\therefore \alpha_1 = 34.72^{\circ}$$

$$and\ \alpha_2 = 124.72^{\circ}$$

(ii) Maximum shear stress

$$f_{t\,max} = \pm\frac{1}{2}\sqrt{(f_x - f_y)^2 + 4q^2}$$

$$= \pm 42.72\ N/mm^2$$

Maximum shear plane

$$\tan 2\beta \ = \frac{f_y - f_x}{2q} = \frac{70 - 100}{2(40)} = -0.375$$

$$\therefore \beta = -10.28^{\circ}$$

Substitute $\beta = \theta = -10.28^{\circ}$ in equation for shear stress.

$$f_t = \frac{f_y - f_x}{2} \sin 2\theta + q \cos 2\theta$$

$$= \frac{70-100}{2} \sin(2(-10.28) + 40 \cos 2(2(-10.28))$$

$$= 42.72\ N/mm^2$$

$$\therefore \beta_1 = -10.28^{\circ}$$

$$\beta_2 = 79.72^{\circ}$$

(iii) Normal stress on a plane at 60° to the vertical plane

$$f_n = \frac{f_x + f_y}{2} + \frac{f_x - f_y}{2} \cos 2\theta + q \sin 2\theta$$

$$= 85 + 15 \cos 120 + 40 \sin 120$$

$$= 112.14\ MPa$$

$$f_t = \frac{f_y - f_x}{2} \sin 2\theta + q \cos 2\theta$$

$$= -15 sin 120 + 40 \cos 120$$

$$= -33\ N/mm^2$$

Graphical method

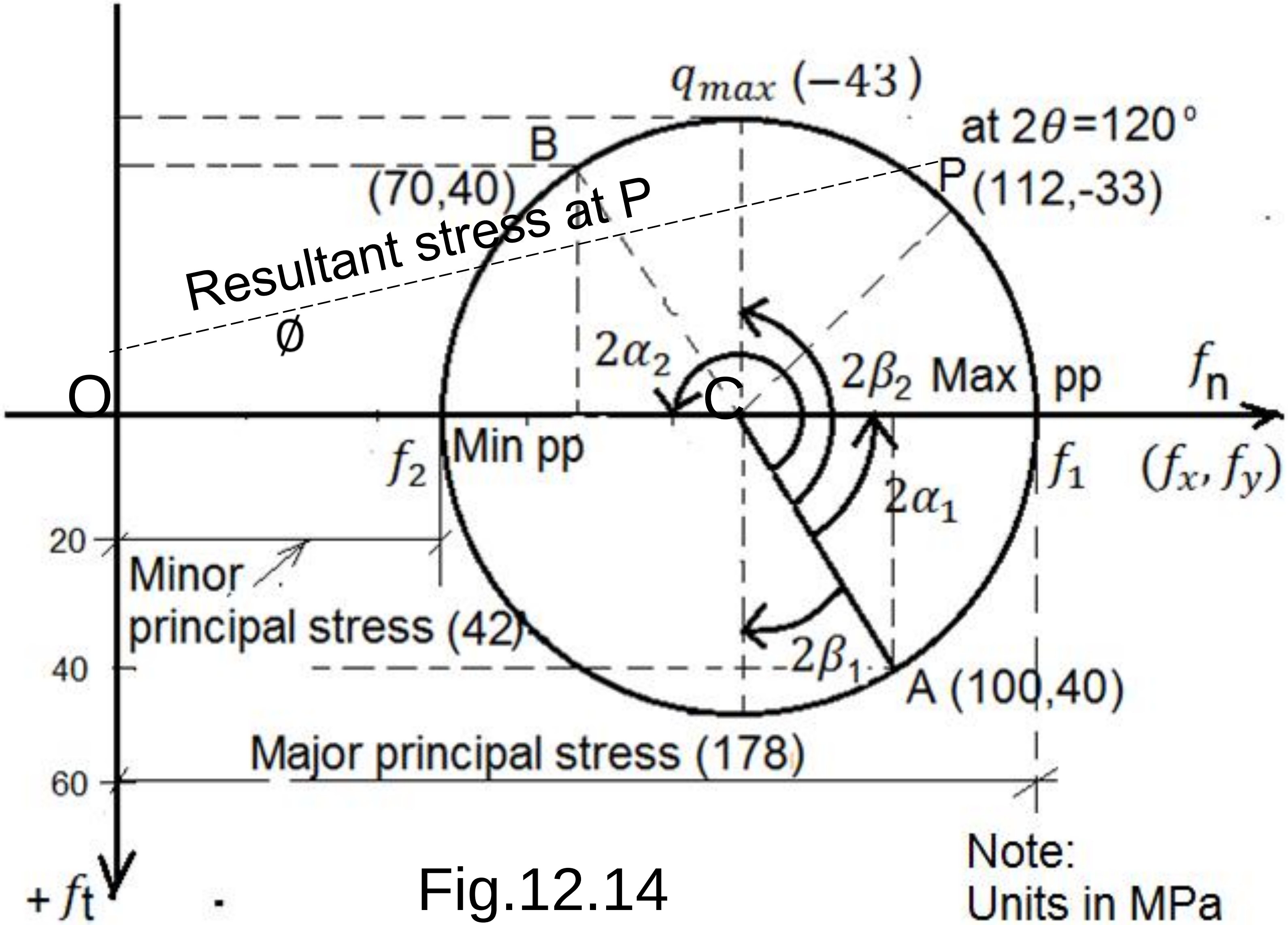


Fig.12.14

From graphical method

Major principal stress, $f_1 = 178\ MPa$

Minor principal stress, $f_2 = 42\ MPa$

Orientation of major PP, $\alpha_1 = 35^0$

Orientation of major PP, $\alpha_2 = 125^0$

Maximum Shear stress, $q_{max} = \pm 43\ MPa$

Orientation of Max shear plane, $\beta_1 = -10$ or

$\beta_2 = 80$

Normal stress @ $60^0 = 112\ MPa$

Shear stress @ $60^0 = -33\ MPa$

STRAIN ENERGY DUE TO BENDING

We know,

strain energy $U = \frac{1}{2} M.\theta$

$$= \frac{1}{2}.M. \int \frac{Mdx}{EI}$$

$$= \int \frac{M^2 dx}{2EI}$$

(or)

We know ,

Strain energy due to axial load,

$$U = \frac{f^2}{2E} \times volume$$

For strain energy stored due to bending, if we substitute

$$f = \frac{M}{I} . y$$

We get,

$$U = \frac{(\frac{M}{I} . y)^2}{2E} . \int A dx$$

$$= \frac{M^2}{2EI^2} Ay^2 \int dx$$

$$= \frac{M^2 l}{2EI}$$

$$U = \int \frac{M^2 dx}{2EI}$$

Ex:113 Find the strain energy stored in a simply supported beam subjected to a point load at centre. Take EI = constant throughout

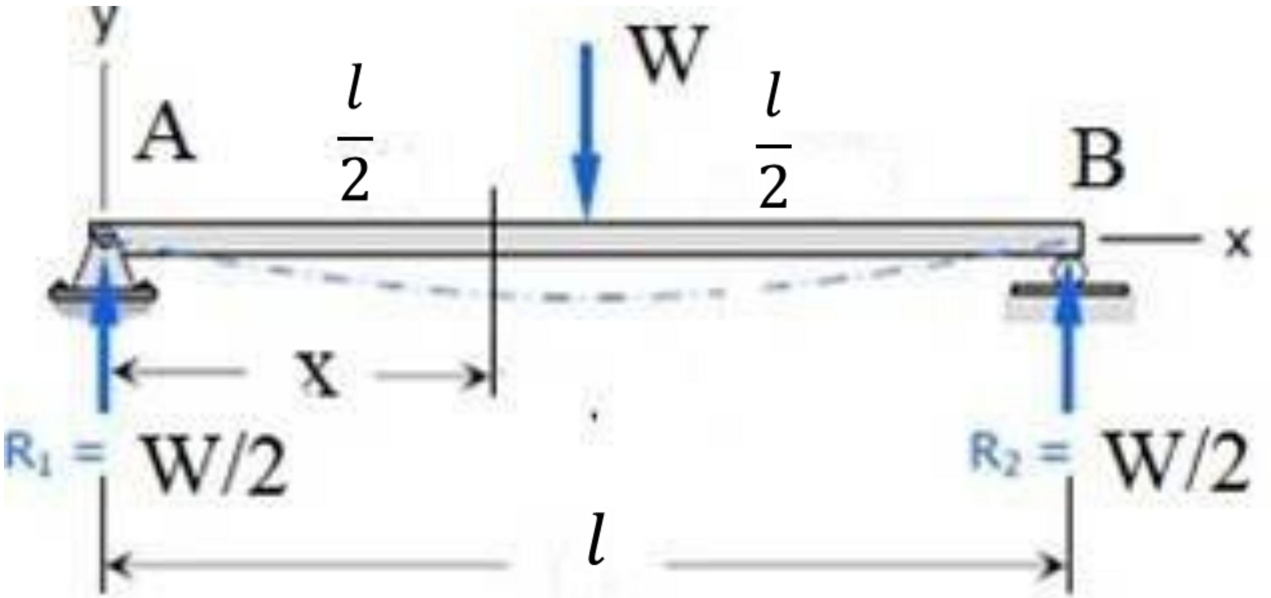


Fig.12.15

Strain energy stored in AC,

$$U_{AC} = \int_0^{\frac{l}{2}} \frac{M^2 dx}{2EI} = \int_0^{\frac{l}{2}} \frac{(\frac{W}{2}x)^2 dx}{2EI}$$

$$= \frac{W^2}{8EI}\left[\frac{x^3}{3}\right]_0^{\frac{l}{2}}$$

$$= \frac{W^2}{24EI}\left(\frac{l}{2}\right)^3 = \frac{W^2 l^3}{128EI}$$

Similarly strain energy stored in BC,

$$U_{BC} = \frac{W^2 l^3}{128EI}$$

$\therefore$ Total strain energy, $U = U_{AC} + U_{BC}$

$$= \frac{W^2 l^3}{96EI}$$

Ex:114 Find the strain energy stored in a simply supported beam subjected to a point load at centre as shown in Fig.12.16

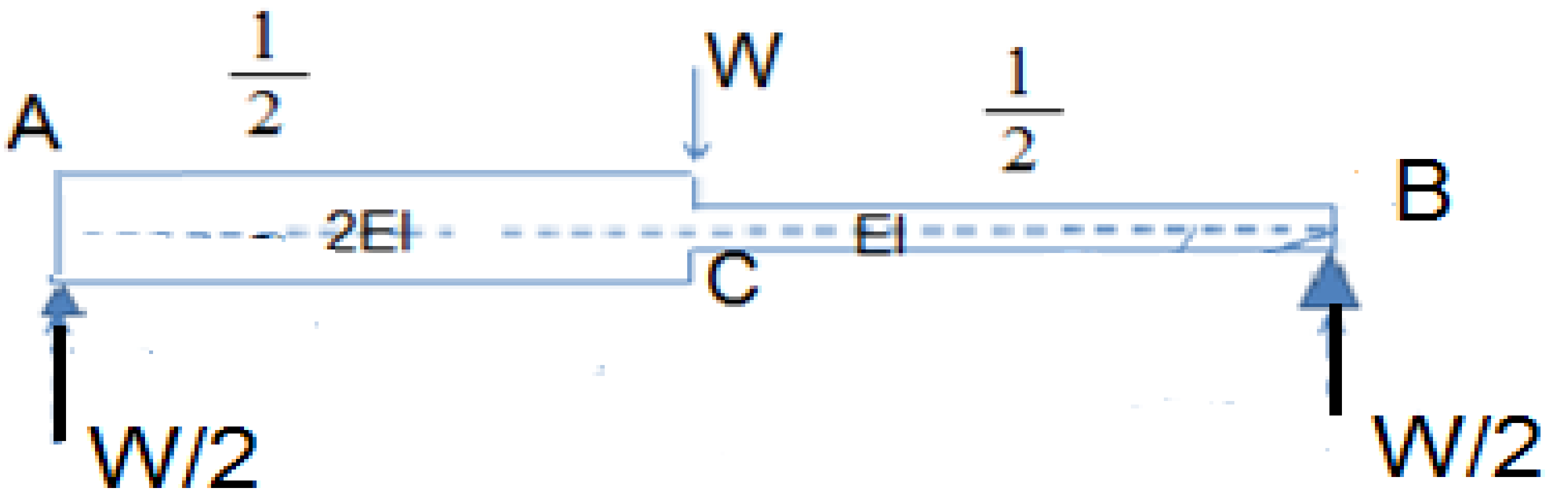


Fig.12.16

Strain energy stored in AC,

$$U_{AC} = \int_0^{\frac{l}{2}} \frac{M^2 dx}{2EI}$$

$$= \int_0^{\frac{l}{2}} \frac{(\frac{W}{2}x)^2 dx}{2(2EI)}$$

$$= \frac{W^2}{16EI}\left[\frac{x^3}{3}\right]_0^{\frac{l}{2}}$$

$$= \frac{W^2}{48EI}(\frac{l}{2})^3$$

$$= \frac{W^2 l^3}{384EI}$$

Strain energy stored in CB,

$$U_{CB} = \int_0^{\frac{l}{2}} \frac{(\frac{W}{2}x)^2 dx}{2EI}$$

$$= \frac{W^2 l^3}{192EI}$$

$\therefore$ Total strain energy stored, $U = U_{AC} + U_{BC}$

$$= \frac{W^2 l^3}{384EI} + \frac{W^2 l^3}{192EI}$$

$$= \frac{W^2 l^3}{128EI}$$

Ex:115. Find the strain energy stored in a simply supported beam as shown in Fig.12.17. Take EI = constant throughout

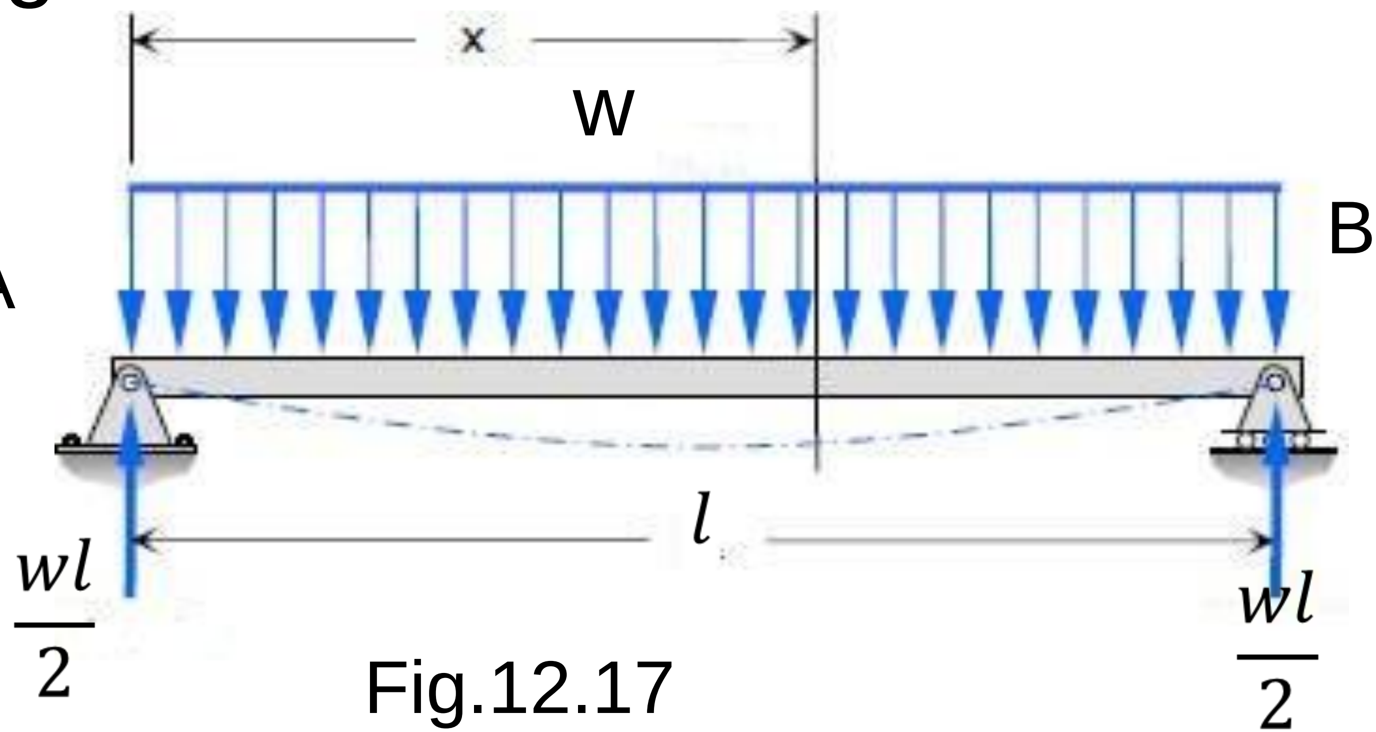


Fig.12.17

Strain energy stored ,$U = \int_0^l \frac{({M_x}^2)dx}{2EI}$

$$= \int_0^l \frac{[\frac{wl}{2}x - \frac{w^2x}{2}]^2}{2EI}dx$$

$$= \frac{w^2}{8EI}\int_0^l (lx - x)^2 dx$$

$$= \frac{w^2}{8EI}\int_0^l (l^2x^2 + x^4 - 2lx^3)dx$$

$$= \frac{w^2}{8EI}\left[l^2\frac{x^3}{3} + \frac{x^5}{5} - 2l\frac{x^4}{4}\right]_0^l$$

$$= \frac{w^2}{8EI}\left[\frac{l^5}{3} + \frac{l^5}{5} - \frac{l^5}{2}\right]$$

$$= \frac{w^2 l^5}{8EI}\left[\frac{10 + 6 - 15}{30}\right]$$

$$= \frac{w^2 l^5}{240EI}$$

Example 12.8: Find the strain energy stored due to bending in a cantilever subjected to a point load at the free end as shown in Fig.12.18. Take EI = constant throughout

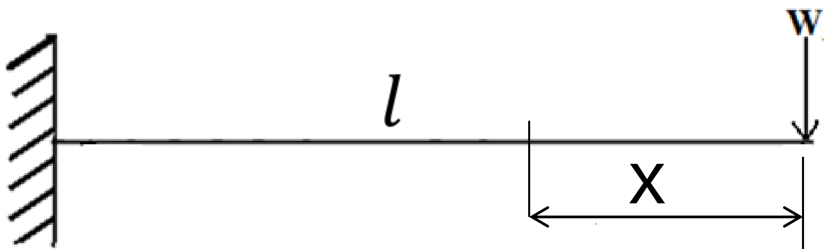


Fig.12.18

$$U = \int_0^l \frac{(Wx)^2 dx}{2EI} = \frac{W^2}{2EI}\left(\frac{l^3}{3}\right) = \frac{W^2 l^3}{6EI}$$

- Ex:117. Find the strain energy stored due to bending in a cantilever subjected to a u.d.l as shown in Fig.12.19 Take EI = constant throughout

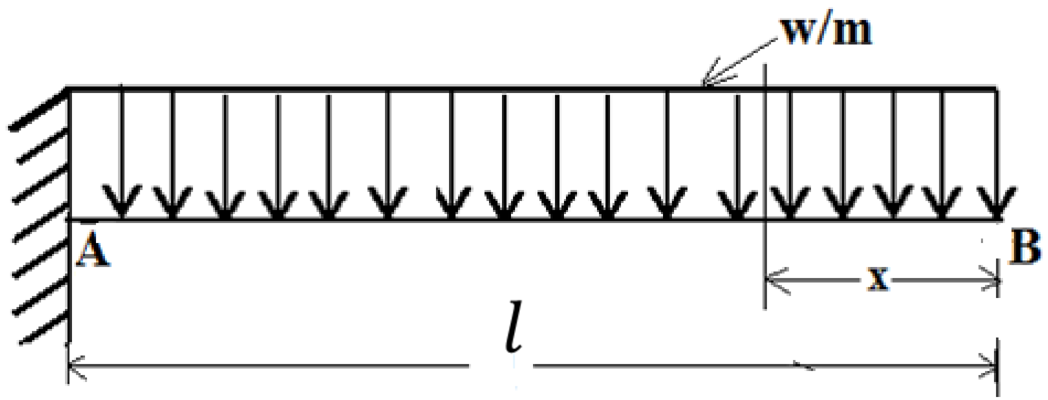


Fig.12.19

$$U = \int_0^l \frac{(\frac{wx^2}{2})^2}{2EI}$$

$$= \frac{w^2}{8EI}\left(\frac{l^3}{3}\right)$$

$$= \frac{W^2 l^3}{24EI}$$